LECTURER NOTES

SUBJECT: Mathematics

BRANCH: Humanities & Science

NAME OF THE FACULTY: SANGHAMITRA NATH

GOVERNMENT POLYTECHNIC, BHADRAK

COURSE OUTCOMES: -

$CO1$- Apply Trigonometry and co-ordinate geometry for the calculation and the mathematical analysis of engineering problems.

$CO2$- Apply the concepts and techniques of complex numbers, partial fraction decomposition, permutations, combinations and the binomial theorem to analyze, simplify, and solve algebraic, combinatorial and mathematical problems efficiently.

$CO3$:- Apply the concepts and operations of matrices and determinants to solve systems of linear equations and analyse mathematical problems arising in science and engineering.

$CO4$:- Apply vector operations to determine projections and angles between vectors, and applying dot and cross products to solve geometric and engineering-related problems in two and three dimensions.

$CO5$:- Apply the concepts of limit to analyse the behaviour of functions, determine continuity, differentiability and solve mathematical and real-world problems involving rates of change and optimization.

$CO6$:- Apply the principles of integral calculus and differential equations to evaluate integrals, determine area under a plane curve, formulate mathematical models and solve first-order differential equations arising in scientific and engineering applications.

Comprehensive Lecture Notes: Trigonometry (Unit 1)

Mathematics (TH-4)

1 Angles and Measurement Systems

1.1 Concept of Angles & Rotational Directions

An angle is generated by rotating a ray around its endpoint from an initial position to a terminal position.

- **Positive Angle:** Formed when the rotation is in the **counter-clockwise** direction.
- **Negative Angle:** Formed when the rotation is in the **clockwise** direction.

1.2 Measurement Systems

Angles can be measured using three primary units:

1. **Sexagesimal System (Degrees):** A complete rotation is divided into $360°$ (degrees).

$$1° = 60' \text{ (minutes)}, \quad 1' = 60'' \text{ (seconds)}$$

2. **Centesimal System (Grades):** A right angle is divided into 100^g (grades).

$$1^g = 100^c \text{ (minutes)}, \quad 1^c = 100^{cc} \text{ (seconds)}$$

3. **Circular System (Radians):** The standard unit of angle subtended at the center of a circle by an arc equal in length to its radius. A full rotation equals 2π radians.

1.3 Conversions Between Units

The foundational relationship linking all three systems is:

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

Where D represents degrees, G represents grades, and R represents radians.

- **Degrees to Radians:** Multiply by $\frac{\pi}{180°}$
- **Radians to Degrees:** Multiply by $\frac{180°}{\pi}$

1.4 Practice Problem Example

Problem: Convert $45°$ into radians and grades.
Solution:

$$R = 45° \times \frac{\pi}{180°} = \frac{\pi}{4} \text{ radians}$$

$$G = \frac{100}{90} \times 45 = 50^g \text{ grades}$$

2 T-Ratios of Allied Angles

2.1 Core Definition and Quadrant Rules

Allied angles are angles like $(-\theta)$, $(90^\circ \pm \theta)$, $(180^\circ \pm \theta)$, $(270^\circ \pm \theta)$, and $(360^\circ \pm \theta)$. Signs are determined by the ASTC rule (All-Silver-Tea-Cups):

- **Quadrant I:** All ratios positive.
- **Quadrant II:** $\sin\theta$ and $\csc\theta$ are positive.
- **Quadrant III:** $\tan\theta$ and $\cot\theta$ are positive.
- **Quadrant IV:** $\cos\theta$ and $\sec\theta$ are positive.

2.2 Trigonometric Transformations

For angles matching $(n \times 90^\circ \pm \theta)$:

- If n is **even**, the ratio stays the same ($\sin \to \sin$).
- If n is **odd**, the ratio switches to its co-function ($\sin \leftrightarrow \cos$, $\tan \leftrightarrow \cot$, $\sec \leftrightarrow \csc$).

$$\sin(90^\circ - \theta) = \cos\theta \quad \sin(90^\circ + \theta) = \cos\theta$$
$$\cos(180^\circ - \theta) = -\cos\theta \quad \tan(180^\circ + \theta) = \tan\theta$$

3 Compound and Transformation Formulae

3.1 Sum and Difference Formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

3.2 Product to Sum/Difference Formulae

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A \sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

3.3 Sum/Difference to Product Formulae

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$
$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$
$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$
$$\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$$

4 Multiple and Sub-Multiple Angle Formulae

4.1 Multiple Angles ($2A$ and $3A$)

$$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$$

4.2 Sub-Multiple Angles ($A/2$)

$$\sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos A}{2}}, \quad \sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{2}}$$

5 Inverse Trigonometric Functions

5.1 Definition and Principal Domains

Inverse trigonometric functions are the inverses of bounded trigonometric functions.

- $\sin^{-1} x = \theta \iff \sin\theta = x$ where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], x \in [-1,1]$
- $\cos^{-1} x = \theta \iff \cos\theta = x$ where $\theta \in [0,\pi], x \in [-1,1]$
- $\tan^{-1} x = \theta \iff \tan\theta = x$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right], x \in \mathbb{R}$

5.2 Key Operational Properties

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad (|x| \le 1)$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \quad (x \in \mathbb{R})$$

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \quad (xy < 1)$$

5.3 Sample Equation Solving Workflow

Problem: Solve for x: $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$.
Solution:

$$\tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right) = \frac{\pi}{4} \implies \frac{5x}{1-6x^2} = \tan\left(\frac{\pi}{4}\right) = 1$$

$$5x = 1 - 6x^2 \implies 6x^2 + 5x - 1 = 0$$

$$(6x-1)(x+1) = 0 \implies x = \frac{1}{6} \text{ or } x = -1$$

Discard $x = -1$ as it violates the principal value constraints. Thus, $x = \frac{1}{6}$.

Co-Ordinate Geometry (Unit 2)

Mathematics (TH-4)

1 Foundations of Coordinate Geometry

1.1 Introduction to Cartesian Co-ordinate Geometry

The Cartesian coordinate system determines each point uniquely in a plane by a pair of numerical coordinates (x, y).

- **X-axis:** The horizontal number line.
- **Y-axis:** The vertical number line intersecting at the origin $(0, 0)$.
- **Quadrants:** The two axes divide the plane into four regions: Quadrant I $(+, +)$, Quadrant II $(-, +)$, Quadrant III $(-, -)$, and Quadrant IV $(+, -)$.

1.2 Distance Formula Derivation and Application

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two distinct points in the Cartesian plane. By constructing a right-angled triangle with the hypotenuse PQ, the horizontal side length is $|x_2 - x_1|$ and the vertical side length is $|y_2 - y_1|$.

Applying the Pythagorean theorem:

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example Application: Find the distance between $A(2, -3)$ and $B(6, 0)$.

$$d = \sqrt{(6 - 2)^2 + (0 - (-3))^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5 \text{ units}$$

1.3 Section Formula (Internal and External Division)

Let a point $R(x, y)$ divide the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $m : n$.

1. **Internal Division:** R lies between P and Q.
$$x = \frac{mx_2 + nx_1}{m + n}, \quad y = \frac{my_2 + ny_1}{m + n}$$
2. **External Division:** R lies on the extension of PQ.
$$x = \frac{mx_2 - nx_1}{m - n}, \quad y = \frac{my_2 - ny_1}{m - n}$$

2 The Straight Line: Slope and Relationships

2.1 Slope of a Line and Inclination Angle

The inclination θ of a line is the angle made by the line with the positive direction of the X-axis measured anti-clockwise $(0 \leq \theta < 180^\circ)$.

- The **Slope** (m) of a line is defined as:
$$m = \tan\theta$$
- For a non-vertical line passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$:
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

2.2 Relationship Between Slopes of Parallel Lines

Two non-vertical lines L_1 and L_2 are parallel if and only if their inclinations are equal ($\theta_1 = \theta_2$).

$$m_1 = m_2$$

2.3 Relationship Between Slopes of Perpendicular Lines

Two non-vertical lines L_1 and L_2 are perpendicular if and only if the product of their slopes equals -1.

$$m_1 \cdot m_2 = -1 \implies m_2 = -\frac{1}{m_1}$$

3 Equations of a Straight Line

3.1 Slope-Intercept and Point-Slope Forms

1. **Slope-Intercept Form:** Given slope m and y-intercept c:

$$y = mx + c$$

2. **Point-Slope Form:** Passed through $P(x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

3.2 Two-Point and Intercept Forms

1. **Two-Point Form:** Passed through $P(x_1, y_1)$ and $Q(x_2, y_2)$:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

2. **Intercept Form:** Line making x-intercept a and y-intercept b:

$$\frac{x}{a} + \frac{y}{b} = 1$$

4 Intersections, Angles, and Distances

4.1 Intersection of Two Straight Lines

To determine the intersection point of two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$, solve the system of equations simultaneously using substitution, elimination, or Cramer's rule.

4.2 Determining the Angle Between Two Straight Lines

If two intersecting lines have slopes m_1 and m_2, the acute angle α between them is found by:

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \quad (1 + m_1 m_2 \neq 0)$$

4.3 Perpendicular Distance From a Point to a Line

The shortest (perpendicular) distance d from a given point $P(x_1, y_1)$ to a line defined by $Ax+By+C = 0$ is:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

5 The Geometry of a Circle

5.1 Definition, Centre, and Radius of a Circle

A circle is the locus of a point that moves in a plane such that its distance from a fixed point (centre) is always constant (radius).

- **Standard Equation:** For a circle with centre (h, k) and radius r:

$$(x-h)^2 + (y-k)^2 = r^2$$

5.2 General Equation of a Circle and Its Features

The general second-degree equation representing a circle is:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

- **Centre:** $(-g, -f)$
- **Radius:** $r = \sqrt{g^2 + f^2 - c}$
- **Existence Condition:** Real circle exists if $g^2 + f^2 - c \geq 0$.

5.3 Diameter Form Equation

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are the endpoints of a diameter of a circle, the equation of the circle is given by:

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

Complex number, Partial fraction, Permutation, Combination and Binomial theorem,Determinants and Matrices (Unit 3)

Mathematics (TH-4)

1 Complex Numbers

1.1 Definition, Real and Imaginary Parts

A complex number z is defined as an ordered expression of the form:

$$z = x + iy$$

where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$ represents the imaginary unit ($i^2 = -1$).

- $\text{Re}(z) = x$ is the **Real Part**.
- $\text{Im}(z) = y$ is the **Imaginary Part**.

1.2 Cartesian Representation & Argand Diagram

A complex number $z = x + iy$ is uniquely represented geometrically as a point $P(x, y)$ on a two-dimensional coordinate system called the **Argand Plane**. The horizontal axis represents the real component, and the vertical axis represents the imaginary component.

1.3 Conjugate and Modulus of a Complex Number

For a complex number $z = x + iy$:

- **Conjugate ($\bar{z}$):** A reflection across the real axis.

$$\bar{z} = x - iy$$

- **Modulus ($|z|$):** Geometric distance from the origin.

$$|z| = \sqrt{x^2 + y^2}$$

1.4 Polar Representation Conversions

A complex number can be expressed in terms of its absolute distance r and counter-clockwise inclination angle θ:

$$z = r(\cos\theta + i\sin\theta)$$

Where the coordinate transformation laws are governed by:

$$r = |z| = \sqrt{x^2 + y^2}, \quad x = r\cos\theta, \quad y = r\sin\theta$$

1.5 Amplitude/Argument across Quadrants

The principal value of the amplitude or argument $\theta = \text{Arg}(z)$ is restricted to $(-\pi, \pi]$ and shifts based on the coordinate sign orientation:

- **Quadrant I ($x > 0, y > 0$):** $\theta = \tan^{-1}\left|\frac{y}{x}\right|$
- **Quadrant II ($x < 0, y > 0$):** $\theta = \pi - \tan^{-1}\left|\frac{y}{x}\right|$

- **Quadrant III** $(x < 0, y < 0)$**:** $\theta = -\pi + \tan^{-1}\left|\frac{y}{x}\right|$
- **Quadrant IV** $(x > 0, y < 0)$**:** $\theta = -\tan^{-1}\left|\frac{y}{x}\right|$

Example: Convert $z = -1 + i\sqrt{3}$ to polar form.
Solution: $x = -1$, $y = \sqrt{3}$ (Quadrant II).

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

$$\theta = \pi - \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

1.6 Basic Arithmetic Operations

Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

- **Addition/Subtraction:** $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$
- **Multiplication:** $z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
- **Division:** Rationalize the denominator by multiplying with its conjugate:

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}$$

1.7 Square Root of a Complex Number

To solve $\sqrt{x + iy} = a + ib$, square both sides to form a simultaneous algebraic system:

$$a^2 - b^2 = x \quad \text{and} \quad 2ab = y$$

Using the identity $a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + (2ab)^2} = \sqrt{x^2 + y^2}$, we isolate values:

$$\sqrt{x + iy} = \pm\left(\sqrt{\frac{|z| + x}{2}} + i \cdot \operatorname{sgn}(y)\sqrt{\frac{|z| - x}{2}}\right)$$

1.8 Cube Roots of Unity

Solving $z^3 = 1$ yields three structural roots: $1, \omega, \omega^2$:

$$1, \quad \omega = \frac{-1 + i\sqrt{3}}{2}, \quad \omega^2 = \frac{-1 - i\sqrt{3}}{2}$$

Fundamental Algebraic Properties:

$$1 + \omega + \omega^2 = 0, \quad \omega^3 = 1$$

1.9 De-Moivre's Theorem and Applications

Statement: For any integer n:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Example Application: Compute $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3$.
Solution: Applying De-Moivre's rules:

$$\cos\left(3 \times \frac{\pi}{6}\right) + i\sin\left(3 \times \frac{\pi}{6}\right) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$$

2 Partial Fractions

2.1 Definitions

An algebraic fraction $P(x)/Q(x)$ is defined as a rational function where $P(x)$ and $Q(x)$ are algebraic polynomials.

- **Proper Fraction:** Degree of $P(x) <$ Degree of $Q(x)$.
- **Improper Fraction:** Degree of $P(x) \geq$ Degree of $Q(x)$. Must be split into a polynomial and a proper remainder fraction via long division before decomposition.

2.2 Decomposition Formats across Configurations

1. **Non-repeated Linear Factors:** $Q(x) = (x-a)(x-b)$
$$\frac{P(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$
2. **Repeated Linear Factors:** $Q(x) = (x-a)^2$
$$\frac{P(x)}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$
3. **Non-repeated Irreducible Quadratic Factors:** $Q(x) = (x-a)(x^2+bx+c)$
$$\frac{P(x)}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$$

Example: Resolve $\frac{3x+1}{(x-1)(x+2)}$ into partial fractions.
Solution: Let $\frac{3x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$
$$3x+1 = A(x+2) + B(x-1)$$
Set $x = 1 \implies 4 = 3A \implies A = \frac{4}{3}$
Set $x = -2 \implies -5 = -3B \implies B = \frac{5}{3}$
$$\frac{3x+1}{(x-1)(x+2)} = \frac{4}{3(x-1)} + \frac{5}{3(x+2)}$$

3 Permutations, Combinations, and Binomial Theorem

3.1 Factorials & Fundamental Counting Principle

- **Factorial:** $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$, where $0! = 1$.
- **Multiplication Principle:** If an operation can occur in m independent ways, and a subsequent operation can occur in n ways, the total composite sequence can occur in $m \times n$ structural ways.

3.2 Permutations vs. Combinations

- **Permutations ($P(n, r)$):** Arrangements where the selection sequence **order matters**.
$$P(n, r) = \frac{n!}{(n-r)!}$$
- **Combinations ($C(n, r)$):** Groups where the selection sequence **order is ignored**.
$$C(n, r) = \frac{n!}{r!(n-r)!}$$

Example Problem: In how many ways can a president and vice-president be chosen from a pool of 5 candidates?
Solution: Order matters, so use permutations:
$$P(5, 2) = \frac{5!}{(5-2)!} = \frac{120}{6} = 20 \text{ structural arrangements}$$

3.3 Binomial Theorem for Positive Integral Index

For any positive integer n:

$$(x + a)^n = C(n, 0)x^n + C(n, 1)x^{n-1}a^1 + C(n, 2)x^{n-2}a^2 + \cdots + C(n, n)a^n$$

3.4 General Term and Middle Term Formulations

- **General Term (T_{r+1}):** Represents the core structure of individual expansion cells:

$$T_{r+1} = C(n, r)x^{n-r}a^r$$

- **Middle Term Determination:**
  - If n is **even**, there is a unique middle term at position:

$$T_{\frac{n}{2}+1}$$

  - If n is **odd**, there are two structural middle terms at positions:

$$T_{\frac{n+1}{2}} \quad \text{and} \quad T_{\frac{n+3}{2}}$$

Example: Find the 4th term in the binomial expansion of $(x + 2)^5$.
Solution: To evaluate T_4, set $r = 3$ and apply the general formula:

$$T_4 = T_{3+1} = C(5, 3)x^{5-3}(2)^3$$

$$C(5, 3) = \frac{5!}{3!2!} = 10 \implies T_4 = 10 \cdot x^2 \cdot 8 = 80x^2$$

4 Introduction to Matrices

4.1 Definition and Order of a Matrix

A matrix is a rectangular array of numbers, symbols, or expressions arranged in horizontal rows and vertical columns. The dimensions or **order** of a matrix are given as $m \times n$, where m is the number of rows and n is the number of columns. An element in the i-th row and j-th column is denoted by a_{ij}.

4.2 Types of Matrices

- **Row Matrix:** A matrix consisting of exactly one row ($1 \times n$).
- **Column Matrix:** A matrix consisting of exactly one column ($m \times 1$).
- **Square Matrix:** A matrix with an equal number of rows and columns ($m = n$).
- **Diagonal Matrix:** A square matrix where all entries outside the main diagonal are zero ($a_{ij} = 0$ for all $i \neq j$).
- **Identity Matrix (I):** A diagonal matrix whose main diagonal elements are all equal to 1.

4.3 Basic Matrix Operations

1. **Addition and Subtraction:** Matrices must be of the same order. Add or subtract corresponding entries:

$$[A \pm B]_{ij} = a_{ij} \pm b_{ij}$$

2. **Scalar Multiplication:** Multiply every entry of the matrix by a scalar scalar k:

$$k \cdot A = [k \cdot a_{ij}]$$

4.4 Matrix Multiplication

Condition for Multiplication: The product of two matrices A and B, denoted as AB, is defined if and only if the number of columns in A equals the number of rows in B. If A is of order $m \times p$ and B is of order $p \times n$, the resulting product $C = AB$ has the order $m \times n$.

Operation Rule: The entry c_{ij} is computed by taking the dot product of the i-th row of A and the j-th column of B:

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Numerical Practice Example: Find AB given:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution: $AB = \begin{bmatrix} (1)(5) + (2)(7) & (1)(6) + (2)(8) \\ (3)(5) + (4)(7) & (3)(6) + (4)(8) \end{bmatrix} = \begin{bmatrix} 5 + 14 & 6 + 16 \\ 15 + 28 & 18 + 32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

5 Determinants

5.1 Introduction and 2nd Order Expansion

A determinant is a scalar value computed from a square matrix. For a 2×2 matrix, its determinant is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

5.2 Expansion of 3rd Order Determinants

A 3×3 determinant is evaluated by expanding along its first row:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

5.3 Structural Properties of Determinants

- **Property 1:** If any two rows or columns of a determinant are identical, its value is zero.
- **Property 2:** Interchanging any two rows or columns changes the sign of the determinant.
- **Property 3:** The value of a determinant remains unchanged if its rows and columns are interchanged ($\det(A) = \det(A^T)$).
- **Property 4:** If all entries of a row or column are multiplied by a scalar scalar k, the value of the determinant is multiplied by k.

5.4 Minors and Co-factors

- **Minor (M_{ij}):** The determinant of the sub-matrix left after deleting the i-th row and j-th column.
- **Co-factor (A_{ij}):** The minor signed according to its positional index:

$$A_{ij} = (-1)^{i+j} M_{ij}$$

5.5 Adjoint and Inverse of a Matrix

The **Adjoint** ($\text{adj}(A)$) is the transpose of the co-factor matrix. The **Inverse** (A^{-1}) exists if $\det(A) \neq 0$:

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

6 Systems of Linear Equations

6.1 Cramer's Rule

Cramer's rule uses determinants to solve a linear system.

- **2-Variable Solutions:** For $a_1x + b_1y = d_1$ and $a_2x + b_2y = d_2$:

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D} \quad \text{(where } D \neq 0\text{)}$$

- **3-Variable Solutions:** For a 3-variable system, variables are isolated via:

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}$$

6.2 Matrix Inversion Method

A linear system can be modeled as the matrix equation $AX = B$. If A^{-1} exists, the variable vector is solved by multiplying the constant column with the inverse matrix:

$$X = A^{-1}B$$

Example System: Solve using the matrix inversion method:

$$x + 2y = 5$$
$$3x + 4y = 11$$

$$\textbf{Solution: } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$|A| = (1)(4) - (2)(3) = -2$$

$$\text{adj}(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \implies A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}\begin{bmatrix} 5 \\ 11 \end{bmatrix} = \begin{bmatrix} (-2)(5) + (1)(11) \\ (1.5)(5) + (-0.5)(11) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hence, $x = 1$ and $y = 2$.

Vector Algebra (Unit 4)

Mathematics (TH-4)

1 Introduction to Vectors

1.1 Vector and Scalar Quantities with Real Examples

- **Scalar Quantities:** Physical quantities that possess only magnitude and no specific direction.
  - *Examples:* Mass (55 kg), Temperature (37°C), Time (10 seconds), Work (200 Joules).
- **Vector Quantities:** Physical quantities that possess both a magnitude and a defined direction, and obey the laws of vector addition.
  - *Examples:* Displacement (10 km North), Velocity (60 km/h East), Force (50 N acting vertically downwards), Weight.

1.2 Notation and Rectangular Resolution of a Vector

A vector is geometrically represented by a directed line segment $\vec{AB}$ or denoted by a bold letter **a** or $\vec{a}$. In a three-dimensional Cartesian coordinate system, any vector can be uniquely resolved into three mutually perpendicular components along the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ (parallel to the x, y, and z axes respectively):

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

The absolute magnitude (length) of the vector is computed via:

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

1.3 Types of Vectors

- **Zero Vector ($\vec{0}$):** A vector whose magnitude is exactly zero. Its initial and terminal points coincide, meaning its direction is indeterminate.
- **Unit Vector ($\hat{a}$):** A vector whose magnitude equals exactly 1 unit. It specifies direction:

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

- **Co-initial Vectors:** Two or more distinct vectors that share the exact same initial starting point.
- **Parallel Vectors:** Vectors whose lines of action are parallel to one another. They can be *like* (same direction) or *unlike* (opposite directions).

1.4 Basic Vector Arithmetic

Given two vectors $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$:

- **Addition and Subtraction:** Performed by combining corresponding components:

$$\vec{a} \pm \vec{b} = (a_x \pm b_x)\hat{i} + (a_y \pm b_y)\hat{j} + (a_z \pm b_z)\hat{k}$$

- **Scalar Multiplication:** Multiplying a vector by a scalar value $m \in \mathbb{R}$ alters its magnitude and shifts direction by 180° if m is negative:

$$m\vec{a} = (ma_x)\hat{i} + (ma_y)\hat{j} + (ma_z)\hat{k}$$

2 Vector Multiplication and Products

2.1 Scalar (Dot) Product of Two Vectors

The scalar or dot product of two vectors $\vec{a}$ and $\vec{b}$ yields a scalar value and is defined as:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

where θ is the angle between the two vectors ($0 \leq \theta \leq \pi$).

Component Form:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

Since $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$.

2.2 Vector (Cross) Product of Two Vectors

The vector or cross product of two vectors $\vec{a}$ and $\vec{b}$ yields a new vector perpendicular to both, defined as:

$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin\theta)\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$, directed by the right-hand thumb rule.

Determinant Form Formulation:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\hat{i} - (a_x b_z - a_z b_x)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$

2.3 Determining the Angle Between Two Vectors

The angle θ between two vectors can be isolated using the dot product formula:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}}$$

- If $\vec{a} \cdot \vec{b} = 0$, the two non-zero vectors are strictly **perpendicular** ($\theta = 90^\circ$).
- If $\vec{a} \times \vec{b} = \vec{0}$, the two non-zero vectors are strictly **parallel** ($\theta = 0^\circ$ or 180°).

2.4 Scalar and Vector Projections

- **Scalar Projection of $\vec{a}$ on $\vec{b}$:** The length of the shadow of $\vec{a}$ cast onto line $\vec{b}$:

$$\text{Scalar Projection} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

- **Vector Projection of $\vec{a}$ on $\vec{b}$:** The scalar projection multiplied by the unit vector $\hat{b}$:

$$\text{Vector Projection} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right)\hat{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

3 Geometric Applications of Vector Products

3.1 Area of a Triangle

If $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a triangle, the area is half the magnitude of their cross product:

$$\text{Area} = \frac{1}{2}|\vec{a} \times \vec{b}|$$

3.2 Area of a Parallelogram

If $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a parallelogram, the total area equals the absolute magnitude of their cross product:

$$\text{Area} = |\vec{a} \times \vec{b}|$$

4 Worked Examples

4.1 Example 1: Dot Product, Angle, and Projections

Problem: Given $\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} + 2\hat{k}$, find $\vec{a} \cdot \vec{b}$, the angle between them, and the scalar projection of $\vec{a}$ on $\vec{b}$.

Solution:

1. **Dot Product:**
$$\vec{a} \cdot \vec{b} = (2)(1) + (1)(3) + (-3)(2) = 2 + 3 - 6 = -1$$

2. **Magnitudes:**
$$|\vec{a}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$
$$|\vec{b}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{1 + 9 + 4} = \sqrt{14}$$

3. **Angle (θ):**
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{-1}{\sqrt{14}\sqrt{14}} = -\frac{1}{14} \implies \theta = \cos^{-1}\left(-\frac{1}{14}\right)$$

4. **Scalar Projection:**
$$\text{Projection} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{-1}{\sqrt{14}}$$

4.2 Example 2: Cross Product and Area Evaluation

Problem: Find the area of a parallelogram whose adjacent sides are given by $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Solution: First, evaluate the vector cross product using a 3rd order matrix determinant expansion:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \hat{i}((-1)(1) - (3)(-7)) - \hat{j}((1)(1) - (3)(2)) + \hat{k}((1)(-7) - (-1)(2))$$
$$\vec{a} \times \vec{b} = \hat{i}(-1 + 21) - \hat{j}(1 - 6) + \hat{k}(-7 + 2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$$

Now, compute its magnitude to evaluate the total area:

$$\text{Area} = |\vec{a} \times \vec{b}| = \sqrt{20^2 + 5^2 + (-5)^2} = \sqrt{400 + 25 + 25} = \sqrt{450} = 15\sqrt{2} \text{ sq. units}$$

Limits and Continuity , Differentiation (Unit 5)

Mathematics (TH-4)

1 Functions and Real Maps

1.1 Definition of a Function, Domain, and Range

A function f from a set A to a set B (denoted $f : A \to B$) is a relation that assigns to each element $x \in A$ exactly one element $y \in B$.

- **Domain:** The set A of all possible inputs for which the function is defined.
- **Range:** The set of all actual outputs $f(x)$ generated as x varies over the domain.

1.2 Elementary Types of Functions

1. **Constant Function:** $f(x) = c$ (where c is a constant). Domain: $\mathbb{R}$, Range: $\{c\}$.
2. **Identity Function:** $f(x) = x$. Domain: $\mathbb{R}$, Range: $\mathbb{R}$.
3. **Absolute Value Function:** $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$. Domain: $\mathbb{R}$, Range: $[0, \infty)$.
4. **Greatest Integer Function:** $f(x) = [x]$, representing the greatest integer less than or equal to x.
5. **Trigonometric Functions:** Maps wrapping coordinates over periodic ratios ($\sin x$, $\cos x$, $\tan x$).
6. **Exponential Function:** $f(x) = a^x$ or e^x $(a > 0, a \neq 1)$. Domain: $\mathbb{R}$, Range: $(0, \infty)$.
7. **Logarithmic Function:** $f(x) = \log_a x$ $(x > 0)$. Domain: $(0, \infty)$, Range: $\mathbb{R}$.

2 Limits and Continuity

2.1 Definition and Existence of a Limit

The limit of a function $f(x)$ as x approaches a point a tells us what value $f(x)$ approaches near a.

- **Left-Hand Limit (LHL):** $\lim_{x \to a^-} f(x)$
- **Right-Hand Limit (RHL):** $\lim_{x \to a^+} f(x)$

Condition for Existence: A limit $\lim_{x \to a} f(x) = L$ exists if and only if both one-sided limits are real, finite, and identical:

$$\text{LHL} = \text{RHL} = L$$

2.2 Algebra of Limits

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then:

$$\lim_{x \to a} [f(x) \pm g(x)] = L \pm M, \quad \lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M, \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad (M \neq 0)$$

2.3 Standard Formulas for Algebraic & Trigonometric Limits

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, \quad \lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{\tan x}{x} = 1, \quad \lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

2.4 Testing Point Continuity

A function $f(x)$ is continuous at a point $x = a$ if it satisfies the following three-part checklist:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

Example: Evaluate $\lim_{x\to 2} \frac{x^3-8}{x-2}$.
Solution: Factoring the numerator using the difference of cubes:

$$\lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x \to 2}(x^2+2x+4) = 2^2 + 2(2) + 4 = 12$$

3 Differentiation Mechanics

3.1 Introduction to Derivatives

The derivative of a function represents its instantaneous rate of change and corresponds geometrically to the slope of the tangent line to the curve at that point. It is mathematically defined using the first principle limit:

$$f'(x) = \frac{d}{dx}[f(x)] = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

3.2 Standard Derivative Rules by Function Group

- **Algebraic Functions:** $\frac{d}{dx}(x^n) = nx^{n-1}$
- **Exponential Functions:** $\frac{d}{dx}(e^x) = e^x, \quad \frac{d}{dx}(a^x) = a^x \ln a$
- **Logarithmic Functions:** $\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
- **Trigonometric Functions:**

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x, \quad \frac{d}{dx}(\tan x) = \sec^2 x$$

- **Inverse Trigonometric Functions:**

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

3.3 Core Operational Rules

Given differentiable functions u and v:

- **Sum and Difference Rule:** $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$
- **Product Rule:** $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$
- **Quotient Rule:** $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- **Chain Rule (Function of a Function):** If $y = f(u)$ and $u = g(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

4 Advanced Differentiation Methods

4.1 Logarithmic Differentiation

Used when handling functions where a variable is in both the base and the exponent, like $y = [f(x)]^{g(x)}$.
Method: Take the natural logarithm (ln) of both sides first to pull down the exponent using properties of logs, then differentiate implicitly.

4.2 Implicit Variations

When x and y are mixed together in an equation and cannot easily be separated into $y = f(x)$, differentiate every term with respect to x, applying the chain rule to terms containing y (multiplying by $\frac{dy}{dx}$), and then isolate $\frac{dy}{dx}$.

4.3 Differentiation of a Function with Respect to Another Function

To find the derivative of a function $u = f(x)$ with respect to another function $v = g(x)$, compute their individual derivatives with respect to x first, then divide them:

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

5 Worked Examples

5.1 Example 1: Quotient Rule and Chain Rule combined

Problem: Differentiate $y = \sin(x^2 + 5)$ with respect to x.
Solution: Let $u = x^2 + 5$, making $y = \sin(u)$. Applying the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u) \cdot (2x) = 2x \cos(x^2 + 5)$$

5.2 Example 2: Derivative of a Function w.r.t Another

Problem: Differentiate $\tan^{-1} x$ with respect to $\sin^{-1} x$.
Solution: Let $u = \tan^{-1} x$ and $v = \sin^{-1} x$.

$$\frac{du}{dx} = \frac{1}{1 + x^2}, \quad \frac{dv}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\frac{1}{1+x^2}}{\frac{1}{\sqrt{1-x^2}}} = \frac{\sqrt{1 - x^2}}{1 + x^2}$$

Integral Calculus and Differential Equation (Unit 6)

Mathematics (TH-4)

1 Indefinite Integral Calculus

1.1 Integration as Inverse Process of Differentiation

Integration is the mathematical inverse operation of differentiation, commonly referred to as finding the antiderivative. If $\frac{d}{dx}[F(x)] = f(x)$, then the indefinite integral of $f(x)$ with respect to x is given by:

$$\int f(x)\,dx = F(x) + c$$

where $c \in \mathbb{R}$ represents the arbitrary constant of integration, accounting for the vertical translation parameter lost during differentiation.

1.2 Standard Elementary Integration Formulas

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1), \qquad \int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^x\,dx = e^x + c, \qquad \int a^x\,dx = \frac{a^x}{\ln a} + c$$

$$\int \sin x\,dx = -\cos x + c, \qquad \int \cos x\,dx = \sin x + c, \qquad \int \sec^2 x\,dx = \tan x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c, \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

1.3 Integration by Substitution Method

When an integrand takes the form $f(g(x))g'(x)$, change the variable by setting $u = g(x)$, meaning $du = g'(x)dx$. This transforms the expression into a standard foundational formula:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Example (Substitution): Evaluate $\int 2xe^{x^2}\,dx$.
Solution: Let $u = x^2 \implies du = 2x\,dx$. Substituting these components:

$$\int e^u\,du = e^u + c = e^{x^2} + c$$

1.4 Integration by Parts & The ILATE Rule

Derived from the product rule of differentiation, the formula for integrating the product of two functions is:

$$\int u\,dv = uv - \int v\,du \implies \int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int \left[f'(x)\int g(x)\,dx\right]dx$$

The selection priority for the first function (u) is governed by the **ILATE** hierarchy: **I**nverse Trigonometric $\to$ **L**ogarithmic $\to$ **A**lgebraic $\to$ **T**rigonometric $\to$ **E**xponential.

Example (Integration by Parts): Evaluate $\int x \ln x\,dx$.
Solution: According to the ILATE rule, let $u = \ln x$ (Logarithmic) and $dv = x\,dx$ (Algebraic). Then $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$.

$$\int x\ln x\,dx = (\ln x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx = \frac{x^2\ln x}{2} - \frac{1}{2}\int x\,dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + c$$

1.5 Integration Using Partial Fractions (Linear Factors)

When the integrand is a rational fraction with non-repeated linear factors in the denominator, decompose it before integrating:

$$\int \frac{px+q}{(x-a)(x-b)}\,dx = \int \left(\frac{A}{x-a} + \frac{B}{x-b}\right) dx = A\ln|x-a| + B\ln|x-b| + c$$

2 Definite Integrals and Geometric Applications

2.1 Fundamental Theorem and Properties

If $F'(x) = f(x)$ on an interval $[a, b]$, the definite integral calculates the accumulation over that interval:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Key Operational Properties:

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
2. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
3. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
4. $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even } (f(-x) = f(x)) \\ 0 & \text{if } f(x) \text{ is odd } (f(-x) = -f(x)) \end{cases}$

2.2 Area Evaluation Bounded by a Curve

The area A enclosed by a continuous curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by:

$$A = \int_a^b |f(x)|\,dx$$

3 Ordinary Differential Equations (ODEs)

3.1 Order and Degree Definitions

- **Order:** The order of the highest-order derivative appearing in the differential equation.
- **Degree:** The power of the highest-order derivative, after the equation has been cleared of fractional radicals and fractional exponents relative to its derivatives.

$$\text{Equation: } \left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y = 0 \implies \text{Order} = 2,\ \text{Degree} = 3$$

3.2 Formation of Differential Equations

To form a differential equation from a geometric family of curves, differentiate the given primitive equation with respect to x as many times as there are independent arbitrary constants, then eliminate those constants.

3.3 Variable Separable Method

If a first-order, first-degree differential equation can be arranged such that all terms in x match with dx and all terms in y match with dy, it can be solved by direct integration:

$$\frac{dy}{dx} = g(x)h(y) \implies \int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

Example: Solve $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$.
Solution: Separating the variables:

$$\int \frac{1}{1+y^2}\,dy = \int \frac{1}{1+x^2}\,dx \implies \tan^{-1} y = \tan^{-1} x + c$$

3.4 First-Order Linear Differential Equations

A standard first-order linear differential equation takes the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

To solve, compute the **Integrating Factor (I.F.)**:

$$\text{I.F.} = e^{\int P(x)\,dx}$$

The general solution is then found using the template formula:

$$y \cdot (\text{I.F.}) = \int Q(x) \cdot (\text{I.F.})\,dx + c$$

Example: Solve $\frac{dy}{dx} + \frac{1}{x}y = x^2$.
Solution: Here $P(x) = \frac{1}{x}$ and $Q(x) = x^2$.

$$\text{I.F.} = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

Multiplying through by the I.F. gives the general solution formula:

$$y \cdot x = \int (x^2 \cdot x)\,dx \implies xy = \int x^3\,dx \implies xy = \frac{x^4}{4} + c \implies y = \frac{x^3}{4} + \frac{c}{x}$$