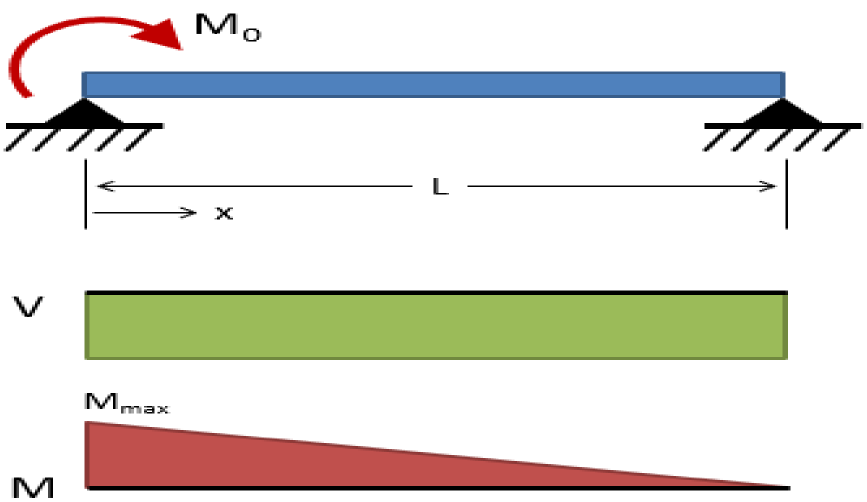
GOVERNMENT POLYTECHNIC, BHADRAK

Strength of Material

(Th- 02)

(As per the 2019-20 syllabus of the SCTE&VT, Bhubaneswar, Odisha)



Third Semester

Mechanical Engg.

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STRENGTH OF MATERIAL

TOPIC WISE DISTRIBUTION PERIODS

Sl no	topics	No. periods as per syllabus	No of periods actually needed	Expected mark	
01	Simple Stress & Strain	10	11	07	08
02	Thin cylindrical and spherical shell under internal pressure	08	09	08	10
03	Two dimensional stress systems	10	11	05	06
04	Bending moment& shear force	10	11		15
05	Theory of simple bending	10	10		14
06	Combined direct & Bending stresses	06	07		12
07	Torsion	06	07		15
	Total	60	67	20	80

CHAPTER-01

SIMPLE STRESS AND STRAIN

LEARNING OBJECTIVES:

Simple stress& strain

Types of load, stresses & strains, (Axial and tangential) Hooke's law, Young's modulus, bulk modulus, modulus of rigidity, Poisson's ratio, derive the relation between three elastic constants,

Principle of super position, stresses in composite section

Temperature stress, determine the temperature stress in composite bar (single core)

Strain energy and resilience, Stress due to gradually applied, suddenly applied and impact load

Simple problems on above.

Introduction:

- The subject strength of materials is basically a study of
  - I. The behaviour of materials under various types of loads and moments.
  - II. The action of forces and their effects on structural and machine elements such as angle irons, circular bars and beams etc.
- The knowledge thus acquired provides rational approach to all design problems, i.e., it helps an engineer to design all types of machines and structures and suggest protective measures for the safe working conditions of such elements. The different structural components may be:
  - I. Trusses, beams and columns of buildings and bridges
  - II. Power transmission shafts, springs and pressure vessels
  - III. Mechanical components in the aircraft, and in the electrical/electronic products

Types Of Load, Stresses & Strains, (Axial And Tangential) Hooke's Law, Young's Modulus, Bulk Modulus, Modulus Of Rigidity, Poisson's Ratio, Derive The Relation Between Three Elastic Constants.

Load:

- A load may be defined as the combined effect of external forces acting on a body.

Classification of Loads:

- The loads may be classified as:
  - i. Dead loads
  - ii. Live or fluctuating loads
  - iii. Inertia loads or forces
  - iv. Centrifugal loads or forces

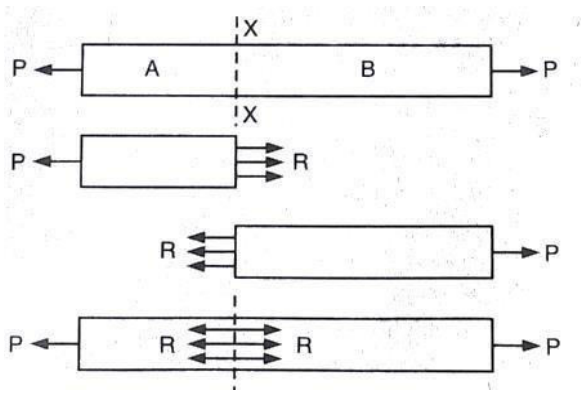
- The other way of classification is
  i. Tensile loads
  ii. Compressive loads
  iii. Torsional or twisting loads
  iv. Bending loads
  v. Shearing loads
- The load may also be a **point (or concentrated)** or **distributed**.

Stress:

- The internal resistance per unit area offered by the material of the body against external loading is called intensity of stress or simply called as stress.

Or

- The internal resistance which the body offers to meet with the load is called stress.
- It is denoted by the symbol 'σ' called sigma.



- Mathematically,

$$\sigma = \frac{R}{A} = \frac{P}{A}$$

Where, σ = stress

R = internal resisting force

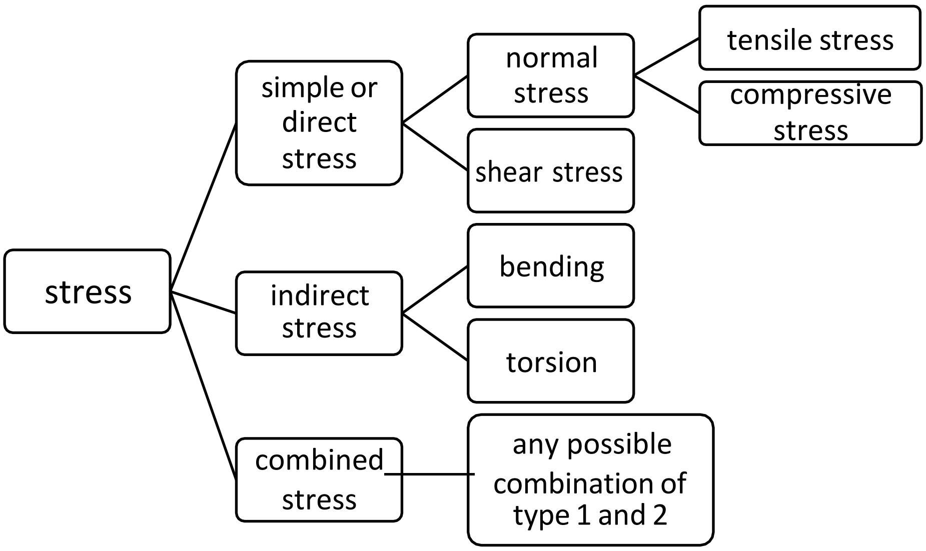
P = Load or external force causing stress to develop

A = area over which stress develops

- Its unit is N/m^2, N/mm^2, KPa, MPa, GPa.
- $1N/m^2$ = 1Pascal, 1Kpa = 10^3 Pascal, 1MPa = 10^6 Pascal, 1GPa = 10^9 Pascal

Classification of stress:

- The various types of stresses may be classified as:

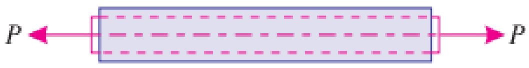


Normal stress:

- The stresses acting normal to the plane on which the forces act are called direct or normal stress.
- The normal stresses are of two types:
    i. Tensile stress
    ii. Compressive stress

Tensile Stress (σ_t):

- When a section is subjected to two equal and opposite axial pulls and the body tends to increase its length, then the stress induced is called tensile stress.

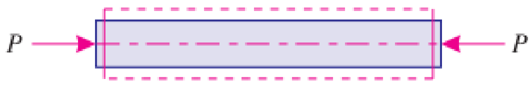


- Mathematically,

$$\sigma_t = \frac{P}{A}$$

Compressive stress (σ_c):

- When a section is subjected to two equal and opposite axial pushes and the body tends to shorten its length, then the stress induced is called compressive stress.

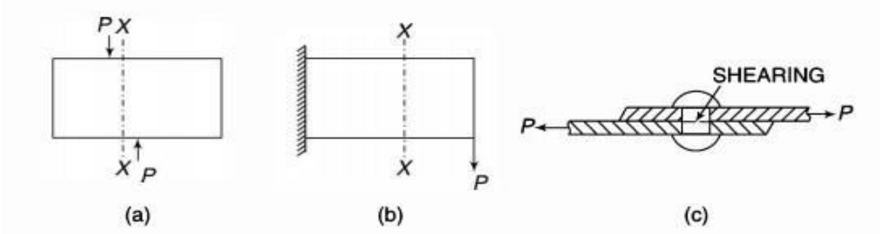


- Mathematically,

$$\sigma_c = \frac{P}{A}$$

Shear Stress:

- When two equal and opposite parallel forces not in the same line act on two parts of a body, then one part tends to slide over or shear from the other across any section and the stress developed is termed as shear stress.
- Shear stress is always tangential to the area over which it acts.
- It is denoted by the symbol 'τ' called 'tau'.



- If P is the force applied and A is the area being sheared, then the intensity of shear stress is given by,

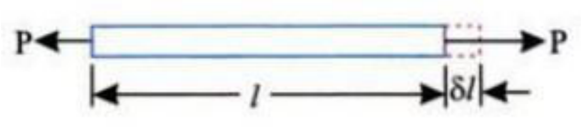
$$r = \frac{P}{A}$$

Strain:

- The strain is the deformation produced by stress.
- The ratio of change in dimension to original dimension of a body is called as strain.
- It is denoted by the letter 'e' or 'ε'.
- It is a unit less quantity.
- Strain(e) = $\frac{change\ in\ dimension}{original\ dimension}$

Tensile strain (e_t):

- A piece of material, with uniform cross section, subjected to a uniform axial tensile stress, will increase its length from $\boldsymbol{l}$ to $\boldsymbol{(l+\delta l)}$ and the increase of length $\boldsymbol{\delta l}$ is the actual deformation of the material.
- It is the ratio of increase in length to the original length of a body.

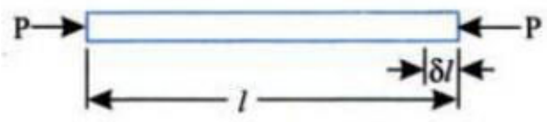


- Mathematically,

$$e_t = \frac{\delta l}{l}$$

Compressive strain (e_c):

- Under compressive forces, a similar piece of material would be reduced in length from $\boldsymbol{l}$ to $\boldsymbol{(l-\delta l)}$.
- It is the ratio of decrease in length to the original length of a body.

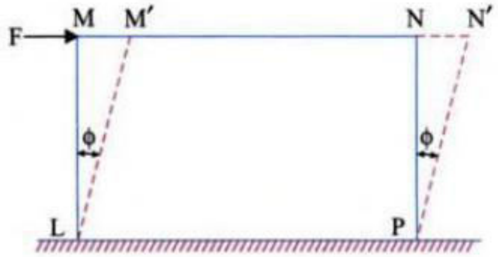


- Mathematically,

$$e_c = \frac{\delta l}{l}$$

Shear strain:

- In case of a shearing load, a shear strain will be produced; this is measured by the angle through which the body distorts.



- Consider a rectangular block LMNP fixed at one face and subjected to force 'F'. After application of force, it distorts through an angle 'ϕ' and occupies new position LM′N′P. the shear strain (e_s) is given by,

$$e_s = \frac{NN'}{NP} = \tan \emptyset$$
$$= \emptyset\ (radians) \ldots \ldots since\ \emptyset\ is\ very\ small$$

- The above result has been obtained by assuming *NN′* equal to arc (as *NN′* is small) drawn with centre *P* and radius *PN*.

Volumetric strain:

- The ratio between change in volume and original volume of the body is called volumetric strain.
- It is denoted by e_v.
- Mathematically,

$$e_v = \frac{change\ in\ volume}{original\ volume} = \frac{\delta v}{v}$$

Elasticity:

- Whenever a body is acted upon by external load, it undergoes some deformation.
- The property by virtue of which the body regains its original shape and size after removal of the external load is called elasticity.

Elastic material:

- If the material regains its original shape and size after removal of the external load, then the material is known as elastic material.

Elastic limit:

There is always a limiting value of load up to which the strain totally disappears on the removal of load. The stress corresponding to this load is called elastic limit.

Hooke's Law:

- Hooke's law states that when a material is loaded within elastic limit, stress is directly proportional to strain.
- Mathematically,

$$stress \text{ a } strain$$
$$\Rightarrow \frac{stress}{strain} = constant\ (E)$$

- Where the constant of proportionality ***E*** is called ***Young's modulus or modulus of elasticity***.

Young's Modulus:

- It is defined as, "The ratio of stress to strain".
- It is denoted by the letter ***'E'***.
- Mathematically,

$$E = \frac{stress}{strain} = \frac{\sigma}{e}$$

- Its unit is same as stress i.e. N/m^2, N/mm^2, KPa, MPa, GPa.

Modulus of Rigidity:

- It is defined as, "The ratio of shear stress to shear stain".
- It is denoted by letter ***C, N* or *G***.
- Mathematically,

$$C = \frac{shear\ stress}{shear\ strain} = \frac{r}{e_s}$$

- Its unit is N/m^2, N/mm^2, KPa, MPa, GPa.

Bulk Modulus or volume modulus of elasticity:

- When a body is subjected to three mutually perpendicular stresses, of equal intensity, the ratio of direct stress to the corresponding volumetric strain is known as bulk modules.
- It is denoted by '***K***'
- Mathematically,

$$K = \frac{direct\ stress}{volumetric\ strain} = \frac{\sigma}{e_v}$$

- Its unit is N/m^2, N/mm^2, KPa, MPa, GPa.

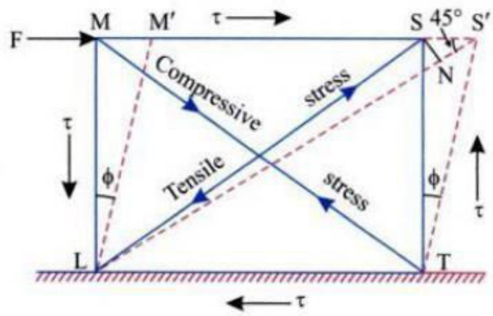
Poisson's Ratio:

- If a body is stressed within its elastic limit, the lateral strain bears a constant ratio to the linear strain this constant is known as Poisson's ratio.
- It is denoted by symbol 'μ' or 1/m.
- It is unit less.
- Mathematically,

$$\mu = \frac{lateral\ strain}{linear\ strain} = \frac{1}{m}$$

  The value of 'μ' lies from 0.25 to 0.34 for different materials.
- Linear strain $= \frac{\sigma}{E}$ and Lateral strain $= \frac{\sigma}{mE}$

Relation between E and C:



Consider a solid cube $LMST$ subjected to a shearing force *'F'*.
Due to shearing load *'F'* let the cube is distorted to $LM'S'T$ and the diagonal LS to LS'.
Let τ be the shear stress produced in the faces MS and LT due to this shearing force.

$$shear\ strain\ (\emptyset) = \frac{SS'}{ST}$$

$$also, shear\ strain = \frac{r}{C}$$

$$\therefore\ \frac{SS'}{ST} = \frac{r}{C} \ldots\ldots\ldots\ldots\ldots\ldots.(i)$$

On the diagonal LS', draw a perpendicular SN from S.

$$Now\ diagonal\ strain = \frac{NS'}{LN} = \frac{NS'}{LS} \ldots\ldots\ldots\ldots(ii)$$

$$NS' = SS' \cos 45^0 = \frac{SS'}{\sqrt{2}}$$

[$\angle LS'T$ is assumed to be equal to $\angle LST$ since SS' is very small.]

$$LS = ST \times \sqrt{2}$$

Putting the value of LS in equation (ii), we get

$$Diagonal\ strain = \frac{SS'}{\sqrt{2}\,ST \times \sqrt{2}} = \frac{SS'}{2ST}$$

$$But, \quad \frac{SS'}{ST} = \frac{r}{C}$$

$$\therefore Diagonal\ strain = \frac{r}{2C} = \frac{\sigma}{2C} \dots\dots\dots\dots (iii)$$

Where, σ is the normal stress due to shear stress (τ).

$$The\ net\ strain\ in\ the\ direction\ of\ diagonal\ LS = \frac{\sigma}{E} + \frac{\sigma}{mE} = \frac{\sigma}{E}[1 + \frac{1}{m}] \dots\dots\dots\dots\dots\dots (iv)$$

[Since the diagonal LS and MT have normal tensile and compressive stress (σ), respectively.]

Comparing equation (iii) and (iv), we get

$$\frac{\sigma}{2C} = \frac{\sigma}{E}[1 + \frac{1}{m}]$$

$$\therefore E = 2C\ [1 + \frac{1}{m}] \dots\dots\dots (v)$$

Relation Between E and K:

It the solid cube is subjected to σ (normal compressive stress) on all the face,

$$The\ direct\ strain\ in\ each\ axis = \frac{\sigma}{E}(compressive)\ and$$

$$lateral\ strain\ in\ other\ axis = \frac{\sigma}{mE}(tensile)$$

$$\therefore Net\ compressive\ strain\ in\ each\ axis = \frac{\sigma}{E} - \frac{\sigma}{mE} - \frac{\sigma}{mE} = \frac{\sigma}{E}[1 - \frac{2}{m}]$$

Volumetric strain (e_v) in each case will be,

$$e_v = 3 \times linear\ strain = 3 \times \frac{\sigma}{E}[1 - \frac{2}{m}]$$

$$But, \quad e_v = \frac{\sigma}{K}$$

$$\therefore \frac{\sigma}{K} = \frac{3\sigma}{E}[1 - \frac{2}{m}] \qquad or \qquad E = 3K\ [1 - \frac{2}{m}] \dots\dots\dots\dots\dots (vi)$$

Relation between E, C & K:

From equation (v) we get,

$$m = \frac{2C}{E - 2C}$$

Substituting the value of m in equation (vi) we have,

$$E = 3K\ [1 - \frac{2}{m}]$$

$$\Rightarrow E = 3K\ [1 - \frac{2}{2C/E - 2C}]$$

$$\Rightarrow E = 3K\ [1 - \frac{E - 2C}{C}]$$

$$\Rightarrow \frac{E}{3K} = [\frac{C - E + 2C}{C}]$$
$$\Rightarrow \frac{E}{3K} = [\frac{3C - E}{C}]$$
$$\Rightarrow \frac{E}{3K} = [\frac{3C}{C} - \frac{E}{C}]$$
$$\Rightarrow \frac{E}{3K} + \frac{E}{C} = 3$$
$$\Rightarrow \frac{EC + 3KE}{3KC} = 3$$
$$\Rightarrow EC + 3KE = 9KC$$
$$\Rightarrow E(3K + C) = 9KC$$
$$\Rightarrow \boldsymbol{E = \frac{9KC}{3K + C}}$$

Principle of super position, stresses in composite section:

Deformation of a body due to force acting on it:

Consider a body subjected to a tensile stress.

Let, P = Load or force acting on the body,

l = Length of the body,

A = Cross-sectional area of the body,

σ = Stress induced in the body,

E = Modulus of elasticity for the material of the body,

e = Strain, and

δl = Deformation of the body

We know that

$$\sigma = \frac{P}{A} \qquad and \qquad e = \frac{\sigma}{E} = \frac{P}{AE}$$

We also know that,

$$e = \frac{\delta l}{l} \Rightarrow \delta l = e \times l$$
$$\Rightarrow \boldsymbol{\delta l = \frac{Pl}{AE}}$$

Notes:

1. The above formula holds good for compressive stress also.

2. For most of the structural materials, the modulus of elasticity for compression is the same as that for tension.

3. Sometimes in calculations, the tensile stress and tensile strain are taken as positive, whereas compressive stress and compressive strain as negative.

***Example 1**: A steel rod 1 m long and 20 mm × 20 mm in cross-section is subjected to a tensile force of 40 kN. Determine the elongation of the rod, if modulus of elasticity for the rod material is 200 GPa.*

SOLUTION. Given : Length (l) = 1 m = 1 × 10^3 mm ; Cross-sectional area (A) = 20 × 20 = 400 mm^2 ; Tensile force (P) = 40 kN = 40 × 10^3 N and modulus of elasticity (E) = 200 GPa = 200 × 10^3 N/mm^2.

We know that elongation of the road,

$$\delta l = \frac{P.l}{A.E} = \frac{(40\times10^3)\times(1\times10^3)}{400\times(20\times10^3)} = 0.5 \text{ mm} \qquad \textbf{Ans.}$$

***Example 2**: A hollow cylinder 2 m long has an outside diameter of 50 mm and inside diameter of 30 mm. If the cylinder is carrying a load of 25 kN, find the stress in the cylinder. Also find the deformation of the cylinder, if the value of modulus of elasticity for the cylinder material is 100 GPa.*

SOLUTION. Given : Length (l) = 2 m = 2 × 10^3 mm ; Outside diameter (D) = 50 mm ; Inside diameter (d) = 30 mm ; Load (P) = 25 kN = 25 × 10^3 N and modulus of elasticity (E) = 100 GPa = 100 × 10^3 N/mm^2.

Stress in the cylinder

We know that cross-sectional area of the hollow cylinder.

$$A = \frac{\pi}{4}\times(D^2 - d^2) = \frac{\pi}{4}\times[(50)^2 - (30)^2] = 1257 \text{ mm}^2$$

and stress in the cylinder,

$$\sigma = \frac{P}{A} = \frac{25\times10^3}{1257} = 19.9 \text{ N/mm}^2 = 19.9 \text{ MPa} \qquad \textbf{Ans.}$$

Deformation of the cylinder

We also know that deformation of the cylinder,

$$\delta l = \frac{P.l}{A.E} = \frac{(25\times10^3)\times(2\times10^3)}{1257\times(100\times10^3)} = 0.4 \text{ mm} \qquad \textbf{Ans.}$$

Principle of superposition:

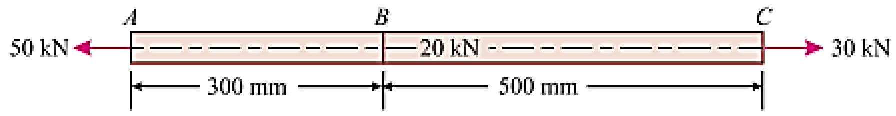
- Sometimes a body is subjected to a number of forces acting on its outer edges as well as at some other sections, at different position along the length of the body. In such a case, the forces are split up and their effects are considered on individual sections. The resulting deformation, of the body, is equal to the algebraic sum of the deformations of the individual sections. This is the principle of superposition which may be stated as:
- **"*The resultant elongation due to several loads acting on a body is the algebraic sum of the elongations caused by individual loads*".**

- Mathematically,

$$\delta l = \frac{P_1 l_1}{AE} + \frac{P_2 l_2}{AE} + \frac{P_3 l_3}{AE} \ldots\ldots\ldots$$

$$= \frac{1}{AE}(P_1 l_1 + P_2 l_2 + P_3 l_3 \ldots\ldots\ldots\ldots\ldots)$$

<u>Example 3</u>: A steel bar of cross-sectional area 200 mm^2 is loaded as shown in Fig. Find the change in length of the bar. Take E as 200 GPa.



SOLUTION. Given: Cross-sectional area (A) = 200 mm^2 and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2.

For the sake of simplification, the force of 50 kN acting at A may be split up into two forces of 20 kN and 30 kN respectively.

Now it will be seen that part AB of the bar is subjected to a tension of 20 kN and AC is subjected to a tension of 30 kN as shown in *Fig. 2.7.

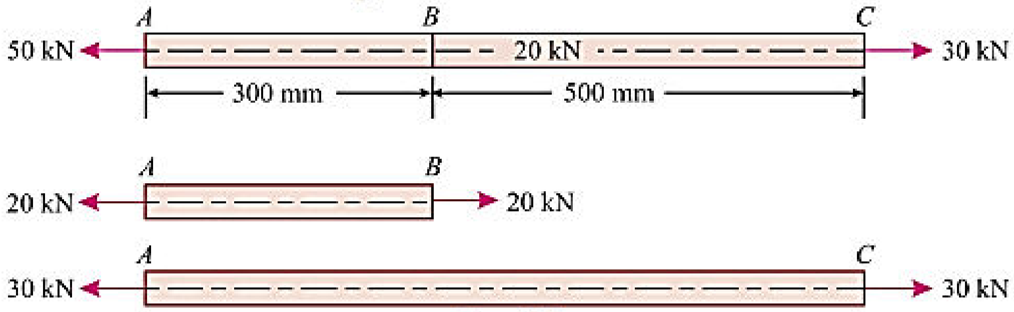


Fig. 2.7

We know that change in length of the bar.

$$\delta l = \frac{1}{AE}(P_1 l_1 + P_2 l_2)$$

$$= \frac{1}{200 \times 200 \times 10^3}\left[[(20\times10^3)\times(300)]+[(30\times10^3)\times(800)]\right] \text{ mm}$$

$$= 0.75 \text{ mm} \qquad \textbf{Ans.}$$

<u>Example 4</u>: A brass bar, having cross-sectional area of 500 mm2 is subjected to axial forces as shown in Fig. Find the total elongation of the bar. Take E = 80 GPa.

SOLUTION. Given: Cross-sectional area (A) = 500 mm^2 and modulus of elasticity (E) = 80 GPa = 80 kN/mm^2.

For the sake of simplification, the force of 100 kN acting at A may be split up into two forces of 80 kN and 20 kN respectively. Similarly, the force of 50 kN acting at C may also be split up into two forces of 20 kN and 30 kN respectively.

Now it will be seen that the part AB of the bar is subjected to a tensile force of 80 kN, part AC is subjected to a tensile force of 20 kN and the part CD is subjected to a compression force of 30 kN as shown in Fig. 2.9.

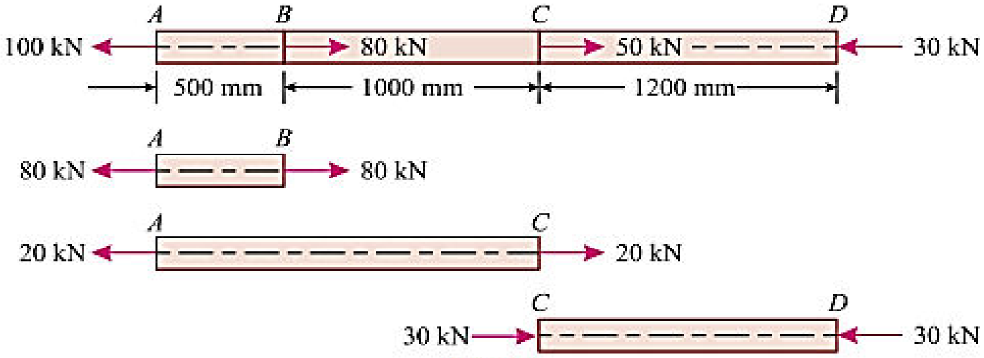


Fig. 2.9

We know that elongation of the bar,

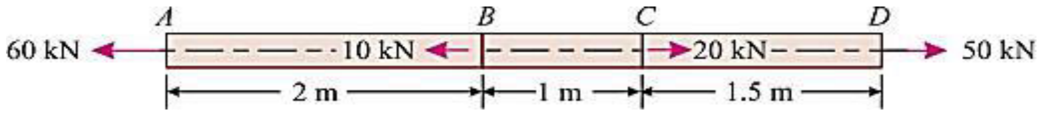
$$\delta l = \frac{1}{AE}\left[P_1 l_1 + P_2 l_2 + P_3 l_3\right]$$

$$= \frac{1}{500 \times 80}\left[(80 \times 500) + (20 \times 1500) - (30 \times 1200\right]\text{mm}$$

...(Taking plus sign for tension and minus for compression)

$$= 0.85 \text{ mm} \qquad \textbf{Ans.}$$

<u>Example 5</u>: A steel rod ABCD 4.5 m long and 25 mm in diameter is subjected to the forces as shown in Fig. If the value of Young's modulus for the steel is 200 GPa, determine its deformation.



Solution. Given: Diameter (D) = 25 mm and Young's modulus (E) = 200 GPa = 200 kN/mm^2.

We know that cross-sectional area of the steel rod.

$$A = \frac{\pi}{4}(D)^2 = \frac{\pi}{4} \times (25)^2 = 491 \text{ mm}^2$$

For the sake of simplification, the force of 60 kN acting at A may be split up into two forces of 50 kN and 10 kN respectively. Similarly the force of 20 kN acting at C may also be split up into two forces of 10 kN and 10 kN respectively.

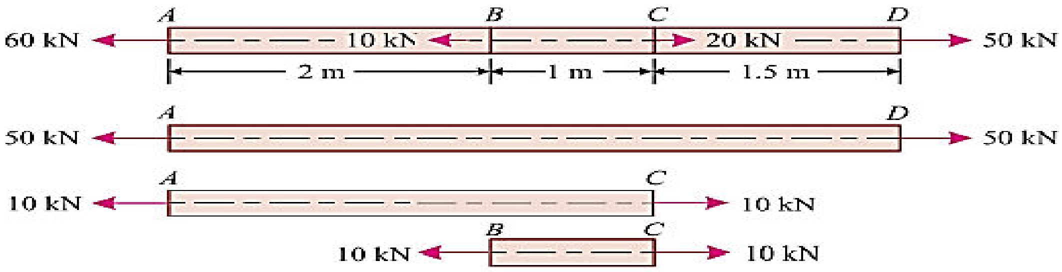


Fig. 2.11

Now it will be seen that the bar AD is subjected a tensile force of 50 kN, part AC is subjected to a tensile force of 10 kN and the part BC is subjected to a tensile force of 10 kN as shown in Fig. 2.11

We know that deformation of the bar,

$$\delta l = \frac{1}{AE}\left[P_1 l_1 + P_2 l_2 + P_3 l_3\right]$$

$$= \frac{1}{491 \times 200}\left[[50 \times (4.5 \times 10^3)] + [10 \times (3 \times 10^3)] + [10 \times (1 \times 10^3)\right]\text{mm}$$

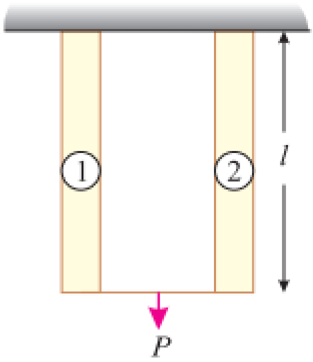
$$= \frac{1}{491 \times 200} \times (265 \times 10^3) = 2.70 \text{ mm} \qquad \textbf{Ans.}$$

Stresses in composite section:

A bar made up of two or more different materials, joined together is called a composite bar. The bars are joined in such a manner, that the system extends or contracts as one unit, equally, when subjected to tension or compression. Following two points should always be kept in view, while solving example on composite bars:

1. Extension or contraction of the bar is equal. Therefore strain (i.e., deformation per unit length) is also equal.
2. The total external load, on the bar, is equal to the sum of the loads carried by the different materials.

Consider a composite bar subjected to load P fixed at the top as shown in figure.



Total load is shared by the two bars, such as:

$$P = P_1 + P_2 = \sigma_1 A_1 + \sigma_2 A_2$$

Further elongations in two bars are same, i.e. strains in the bars are equal. Thus

$$e_1 = e_2 \Rightarrow \frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2}$$

$$\therefore \ \boldsymbol{\frac{\sigma_1}{\sigma_2} = \frac{E_1}{E_2}}$$

The ratio E_1/E_2 is called the modular ratio.

<u>Example 6</u>: A reinforced concrete circular section 50,000 mm^2 cross-sectional area carrying 6 reinforcing bars whose total area is 500mm^2. Find the safe load, the column can carry, if the concrete is not to be stressed more than 3.5Mpa. Take modular ratio for steel and concrete as 18.

<u>Data given</u>:

Area of column (A) = 50,000 mm^2, No of reinforcing bars = 6, Total area of steel bars (A_S) = 500 mm^2, Maximum stress in concrete (σ_C) = 3.5 MPa = 3.5 N/mm^2, Modular ratio (E_S/E_C) = 18.

We know that area of concrete,

$$A_C = 50000 - 500 = 49500\ mm^2$$

And stress in steel,

$$\sigma_S = \frac{E_S}{E_C} \times \sigma_C = 18 \times 3.5 = 63\ N/mm^2$$

∴ Safe load, the column can carry,

$$P = (\sigma_S \,.\, A_S) + (\sigma_C \,.\, A_C) = (63 \times 500) + (3.5 \times 49500) N$$

$$= 204750\ N = 204.75\ kN$$

<u>Example 7</u>: A reinforced concrete column 500mm x 500mm in cross section is reinforced with 4 steel bars of 25 mm diameter, one in each corner. The column is carrying a load of 1000 kN. Find the stresses in the concrete and steel bars. Take E for steel = 200 GPa and E for concrete = 14 GPa.

Data Given:

Area of column = 500 × 500 = 2,50,000 mm^2, No of steel bars (n) = 4, Diameter of steel bars (d) = 25 mm, Load on column (P) = 1000 kN = 1000 × 10^3 N, E_S = 210 GPa, E_C = 14 GPa.

We know that area of steel bars,

$$A_S = 4 \times \frac{\pi}{4} \times d^2\ mm^2$$
$$= 4 \times \frac{\pi}{4} \times (25)^2 = 1963\ mm^2$$

∴ Area of concrete,

$$A_C = 250000 - 1963 = 248037\ mm^2$$

We also know that stress in steel,

$$\sigma_S = \frac{E_S}{E_C} \times \sigma_C = \frac{210}{14} \times \sigma_C = 15\ \sigma_C$$

Total load (P),

$$1000 \times 10^3 = (\sigma_S \,.\, A_S) + (\sigma_C \,.\, A_C)$$
$$\Rightarrow 1000 \times 10^3 = (15\ \sigma_C \times 1963) + (\sigma_C \times 248037) = 277482\ \sigma_C$$
$$\Rightarrow \sigma_C = \frac{1000 \times 10^3}{277482} = 3.6\ N/mm^2 = 3.6\ MPa$$
$$and\ \ \sigma_S = 15\ \sigma_C = 15 \times 3.6 = 54\ MPa$$

Temperature stress, determine the temperature stress in composite bar (single core):

Temperature Stress:

Whenever there is some increase or decrease in the temperature of a body, it causes the body to expand or contract. If the body is allowed to expand or contract freely, with the rise or fall of the temperature, no stresses are induced in the body. But if the deformation of the body is prevented, some stresses are induced in the body. Such stresses are called temperature stresses. The corresponding strains are called temperature strains.

- ***Define temperature stress or thermal stress***.

  Ans: It is defined as the stress produced due to prevention of elongation or contraction of a bar in order to increase or decrease of temperature.

Temperature Stresses in Simple Bars:

Consider a bar of uniform cross-section is subjected to an increase in temperature.

Let, l = Original length of the bar.

t = Increase of temperature

α = Coefficient of linear expansion

The increase in length of the bar due to increase of temperature will be

$$\delta l = l.\alpha.t$$

If this elongation in the bar is prevented by some external force or by fixing the bar ends, the temperature strain (compressive) thus produced will be given by,

$$Temperature\ strain\ (e) = \frac{\delta l}{l} = \frac{l.\alpha.t}{l} = \alpha.t$$

$$\therefore Temperature\ stress\ developed(\sigma) = e.E = \alpha.t.E\ (compressive)$$

If the temperature of the bar is lowered, the temperature strain and stress will be tensile in nature.

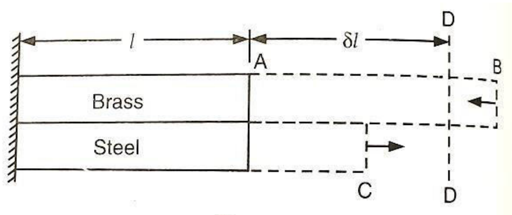
Note:

- The value of coefficient of linear expansion of materials in everyday use are given below

S. No.	Material	Coefficient of linear expansion/°C (α)
1.	Steel	11.5×10^{-6} to 13×10^{-6}
2.	Wrought iron, Cast iron	11×10^{-6} to 12×10^{-6}
3.	Aluminium	23×10^{-6} to 24×10^{-6}
4.	Copper, Brass, Bronze	17×10^{-6} to 18×10^{-6}

Temperature Stresses In Composite Bars:

Consider temperature rise of a composite bar consisting of two different materials, one of steel and other of brass, rigidly fastened to each other.



If allowed to expand freely;

Expansion of brass bar (AB) = $l.\alpha_b.t$

Expansion of steel bar (AC) = $l.\alpha_s.t$

Since coefficient of thermal expansion of brass is greater than that of steel, expansion of brass will be more. But the bars are fastened together and accordingly both will expand to the same final position represented by DD with net expansion of composite system AD equal to δl. To attain this position, brass bar is pushed back and the steel bar is pulled. Obviously compressive stress will be induced in brass bar and tensile stress will be developed in steel bar.

Under equilibrium state,

Compressive force in brass = Tensile force in steel

$\Rightarrow \sigma_b A_b = \sigma_s A_s$

Corresponding to brass bar:

Reduction in elongation,

$$DB = AB - AD = l\ \alpha_b\ t - \delta l$$

$$Strain, e_b = \frac{l\, a_b\, t - \delta l}{l} = a_b t - e$$

Where $e = \delta l/l$ is the actual strain of the composite bar.

Corresponding to steel bar:

Extra elongation,

$$CD = AD - AC = \delta l - l\, \alpha_s\, t$$

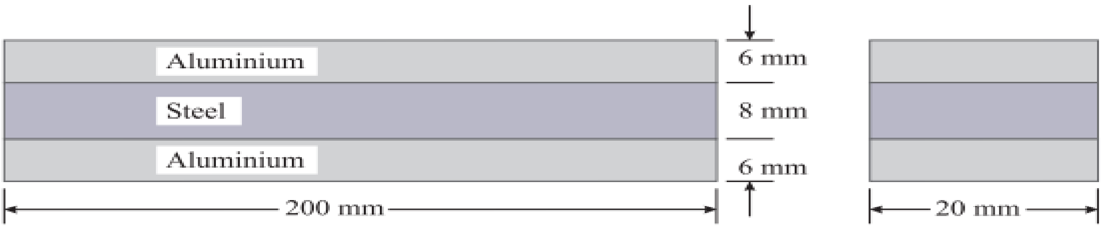
$$Strain, e_s = \frac{\delta l - l\, a_s\, t}{l} = e - a_s t$$

Adding e_b and e_s, we get

$$e_b + e_s = (a_b - a_s)t$$

** It may be noted that the nature of the stresses in the bars will get reversed if there is reduction in the temperature of the composite bar.

<u>Example 8</u>: A flat steel bar 200 mm × 20 mm × 8 mm is placed between two aluminium bars 200 mm × 20 mm × 6 mm so as to form a composite bar as shown in Fig.



All the three bars are fastened together at room temperature. Find the stresses in each bar, Where the temperature of the whole assembly is raised through 50°C. Assume:

Young's modulus for steel = 200 GPa

Young's modulus for aluminium = 80 GPa

Coefficient of expansion for steel = 12 × 10^-6/°C

Coefficient of expansion for aluminium = 24 × 10^-6/°C

Solution. Given : Size of steel bar = 200 mm × 20 mm × 8 mm ; Size of each aluminium bar = 200 mm × 20 mm × 6 mm ; Rise in temperature (t) = 50°C ; Young's modulus for steel (E_S) = 200 GPa = 200×10^3 N/mm^2 ; Young's modulus for aluminium (E_A)= 80 GPa = 80×10^3 N/mm^2; Coefficient of expansion for steel (α_S) = 12×10^{-6}/°C and coefficient of expansion for aluminium (α_A) = 24×10^{-6}/°C.

Let σ_S = Stress in steel bar and

σ_A = Stress in each aluminium bar.

We know that area of steel bar

$$A_S = 20 \times 8 = 160 \text{ mm}^2$$

and total area of two aluminium bars,

$$A_A = 2 \times 20 \times 6 = 240 \text{ mm}^2$$

We also know that when the temperature of the assembly will increase, the free expansion of aluminium bars will be more than that of steel bar (because α_A is more than α_S). Thus the aluminium bars will be subjected to compressive stress and the steel bar will be subjected to tensile stress. Since the tensile load on the steel bar is equal to the compressive load on the aluminium bars, therefore stress in steel bar,

$$\sigma_S = \frac{A_A}{A_S} \times \sigma_A = \frac{240}{160} \times \sigma_A = 1.5\,\sigma_A$$

We know that strain in steel bar,

$$\varepsilon_S = \frac{\sigma_S}{E_S} = \frac{\sigma_S}{200 \times 10^3}$$

and

$$\varepsilon_A = \frac{\sigma_A}{E_A} = \frac{\sigma_A}{80 \times 10^3}$$

We also know that total strain,

$$\varepsilon_S + \varepsilon_A = t\,(\alpha_A - \alpha_S)$$

$$\frac{\sigma_S}{200 \times 10^3} + \frac{\sigma_A}{80 \times 10^3} = 50\,[(24 \times 10^{-6}) - (12 \times 10^{-6})]$$

$$\frac{1.5\sigma_A}{200 \times 10^3} + \frac{\sigma_A}{80 \times 10^3} = 50 \times (12 \times 10^{-6})$$

$$20 \times 10^{-6}\,\sigma_A = 600 \times 10^{-6} \quad \text{or} \quad 20\,\sigma_A = 600$$

$$\therefore \quad \sigma_A = \frac{600}{20} = 30 \text{ N/mm}^2 = 30 \text{ MPa} \quad \textbf{Ans.}$$

and

$$\sigma_S = 1.5\,\sigma_A = 1.5 \times 30 \text{ N/mm}^2 = 45 \text{ MPa} \quad \textbf{Ans.}$$

<u>Example 9</u>: A gun metal rod 20 mm diameter, screwed at the ends, passes through a steel tube 25 mm and 30 mm internal and external diameters respectively. The nuts on the rod are screwed tightly home on the ends of the tube. Find the intensity of stress in each metal, when the common temperature rises by 200°F.

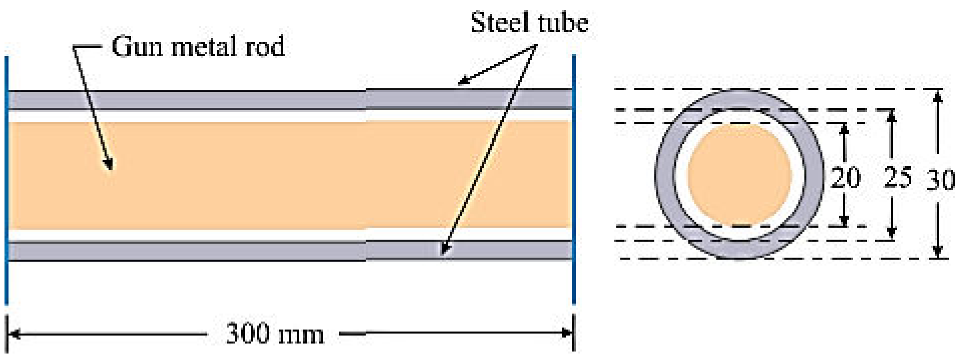
Take: Coefficient of expansion for steel = 6 × 10^{-6}/°F

Coefficient of expansion for gun metal = 10 × 10^{-6}/°F

Modulus of elasticity for steel = 200 GPa

Modulus of elasticity for gun metal = 100 GPa.

SOLUTION. Given : Diameter of gun metal rod = 20 mm ; Internal diameter of steel tube = 25 mm; External diameter of steel tube = 30 mm ; Rise in temperature (t) = 200°F ; Coefficient of expansion for steel (α_S) = 6 × 10^{-6}/°F ; Coefficient of expansion for gun metals (α_G) = 10 × 10^{-6}/°F; Modulus of elasticity for steel (E_S) = 200 GPa = 200 × 10^3 N/mm² and modulus of elasticity for gun metal (E_G) = 100 GPa = 100 × 10^3 N/mm².



Let σ_G = Stress in gun metal rod, and

σ_S = Stress in steel tube,

We know that area of gun metal rod,

$$A_G = \frac{\pi}{4} \times (20)^2 = 100\,\pi \text{ mm}^2$$

and area of steel tube
$$A_S = \frac{\pi}{4} [(30)^2 - (25)^2] = 68.75\,\pi \text{ mm}^2$$

We also know that when the common temperature of the gun metal rod and steel tube will increase, the free expansion of gun metal rod will be more than that of steel tube (because α_G is greater than α_S). Thus the gun metal rod will be subjected to compressive stress and the steel tube will be subjected to tensile stress. Since the tensile load on the steel tube is equal to the compressive load on the gun metal rod, therefore stress in steel,

$$\sigma_S = \frac{A_G}{A_S} \times \sigma_S = \frac{100\pi}{68.75\pi} \times \sigma_G = 1.45\,\sigma_G$$

We know that strain in steel tube,

$$\varepsilon_S = \frac{\sigma_S}{E_S} = \frac{\sigma_S}{200 \times 10^3}$$

and
$$\varepsilon_G = \frac{\sigma_G}{E_G} = \frac{\sigma_G}{100 \times 10^3}$$

We also know that total strain,

$$\varepsilon_S + \varepsilon_G = t\,(\alpha_G - \alpha_S)$$

$$\frac{\sigma_S}{200 \times 10^3} + \frac{\sigma_G}{100 \times 10^3} = 200\,[(10 \times 10^{-6}) - (6 \times 10^{-6})]$$

$$\frac{1.45\sigma_G}{200 \times 10^3} + \frac{\sigma_G}{100 \times 10^3} = 200 \times (4 \times 10^{-6})$$

$$\frac{3.45\sigma_G}{200 \times 10^3} = 800 \times 10^{-6}$$

$$3.45\ \sigma G = (800 \times 10^{-6}) \times (200 \times 10^3) = 160$$

$\therefore$
$$\sigma_G = \frac{160}{3.45} = 46.4 \text{ N/mm}^2 = 46.4 \text{ MPa} \qquad \textbf{Ans.}$$

and
$$\sigma_S = 1.45\,\sigma_G = 1.45 \times 46.4 = 67.3 \text{ MPa} \qquad \textbf{Ans.}$$

Strain energy and resilience, Stress due to gradually applied, suddenly applied and impact load:

Strain Energy:

- When an elastic body is loaded within elastic limit, it deforms and some work is done which is stored within the body in the form of internal energy. This stored energy in the deformed body is known as strain energy or potential energy of deformation and is denoted by ***'U'***.

Resilience:

- It is the ability of a material to regain its original shape on removal of the applied load. It is defined as the strain energy per unit volume in simple tension or compression and is equal to $\boldsymbol{\sigma^2/2E}$. It is also referred as strain energy density and is denoted by ***'u'***.

Stresses due to different types of loads:

A body may be subjected to following types of loads:

1. Gradually applied loads
2. Suddenly applied loads
3. Falling or impact loads

1. Gradually applied loads:

A body is said to be acted upon by a gradually applied load if the load increases from zero and reaches its final value stepwise.

Now consider a metallic bar subjected to a gradual load.

Let, P = Load gradually applied

A = Cross sectional area of the bar

l = Length of the bar

E = Modulus of elasticity of the bar material

δl = Deformation of the bar due to load

Since the load applied is gradual, and varies from zero to P, therefore the average load is equal to P/2

$$\therefore Work\ done = Force \times Distance$$

$$= Average\ load \times Deformation$$

$$= \frac{P}{2} \times \delta l = \frac{P}{2}(e.l)$$

$$= \frac{1}{2}\sigma\,.\,e\,.\,A\,.\,l$$

$$= \frac{1}{2} \times (stress \times strain \times volume)$$

$$= \frac{1}{2} \times \sigma\,.\frac{\sigma}{E}\,.A\,l$$

$$= \frac{1}{2} \times \frac{\sigma^2}{E} \times A\, l$$

$$= \frac{\boldsymbol{\sigma^2}}{\boldsymbol{2E}} \times \boldsymbol{V}$$

Since the energy stored is also equal to the work done, therefore strain energy stored,

$$\boldsymbol{U = \frac{\sigma^2}{2E} \times V}$$

We also know that resilience = strain energy per unit volume

$$= \sigma^2/2E$$

<u>Example 10</u>: Calculate the strain energy stored in a bar 2 m long, 50 mm wide and 40 mm thick when it is subjected to a tensile load of 60kN. Take E as 200 GPa.

SOLUTION. Given : Length of bar (l) = 2 m = 2×10^3 mm ; Width of bar (b) = 50 mm ; Thickness of bar (t) = 40 mm ; Tensile load on bar (P) = 60 kN = 60×10^3 N and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2

We know that stress in the bar,

$$\sigma = \frac{P}{A} = \frac{60 \times 10^3}{50 \times 40} = 30 \text{ N/mm}^2$$

∴ Strain energy stored in the bar,

$$U = \frac{\sigma^2}{2E} \times V = \frac{(30)^2}{2 \times (200 \times 10^3)} \times 4 \times 10^6 \text{ N-mm}$$

$$= 9 \times 10^3 \text{ N-mm} = 9 \text{ kN-mm} \qquad \textbf{Ans.}$$

2. <u>Suddenly applied loads</u>:

When the load is applied all of a sudden and not stepwise is called suddenly applied load.

Now consider a bar subjected to a sudden load.

Let, P = Load applied suddenly

A = Cross sectional area of the bar

l = Length of the bar

E = Modulus of elasticity of the bar material

δl = Deformation of the bar due to load

σ = Stress induced by the application of the sudden load

Since the load is applied suddenly, therefore the load (P) is constant throughout the process of deformation of the bar.

$$\therefore Work\ done = Force \times Distance$$

$$= load \times Deformation$$

$$= P \times \delta l$$

We know that strain energy stored,

$$U = \frac{\sigma^2}{2E} \times A\, l$$

Since the energy stored is equal to work done, therefore

$$\frac{\sigma^2}{2E} \times A\, l = P \times \delta l = P \times \frac{\sigma}{E}\, l$$

$$\Rightarrow \boldsymbol{\sigma = 2 \times \frac{P}{A}}$$

<u>Example 11</u>*: *An axial pull of 20 kN is suddenly applied on a steel rod 2.5 m long and 1000 mm² in cross-section. Calculate the strain energy, which can be absorbed in the rod. Take E = 200 GPa.

SOLUTION. Given : Axial pull on the rod (P) = 20 kN = 20×10^3 N ; Length of rod (l) = 2.5 m = 2.5×10^3 mm ; Cross-sectional area of rod (A) = 1000 mm² and modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm².

We know that stress in the rod, when the load is suddenly applied,

$$\sigma = 2 \times \frac{P}{A} = 2 \times \frac{20 \times 10^2}{1000} = 440 \text{ N/mm}^2$$

and volume of the rod,

$$V = l \,.\, A = (2.5 \times 10^3) \times 1000 = 2.5 \times 10^6 \text{ mm}^3$$

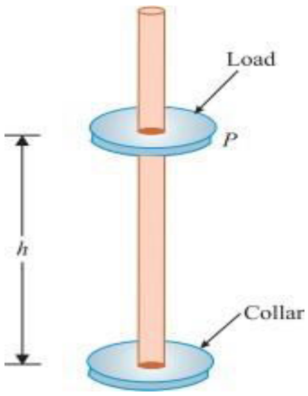
$\therefore$ Strain energy which can be absorbed in the rod,

$$U = \frac{\sigma^2}{2E} \times V = \frac{(40)^2}{2 \times (200 \times 10^3)} \times (2.5 \times 10^6) \text{ N-mm}$$

$$= 10 \times 10^3 \text{ N-mm} = 10 \text{ kN-mm} \qquad \textbf{Ans.}$$

3. <u>Falling or impact loads</u>:

The load which falls from a height or strike the body with certain momentum is called falling or impact load. Now consider a bar subject to a load applied with impact as shown in Figure.



Let, P = Load applied with impact

A = Cross sectional area of the bar

l = Length of the bar

E = Modulus of elasticity of the bar material

δl = Deformation of the bar due to load

σ = Stress induced by the application of this load with impact

h = Height through which the load will fall, before impacting on the collar

$$\therefore Work\ done = Load \times Distance\ moved$$

$$= P \times (h + \delta l)$$

And energy stored,

$$U = \frac{\sigma^2}{2E} \times A\, l$$

Since energy stored is equal to the work done, therefore

$$\frac{\sigma^2}{2E} \times A\,l = P(h + \delta l) = P\,(h + \frac{\sigma}{E}.\,l)$$

$$\Rightarrow \frac{\sigma^2}{2E} \times A\,l = Ph + \frac{P\sigma l}{E}$$

$$\therefore \sigma^2\,(\frac{Al}{2E}) - \sigma\,(\frac{Pl}{E}) - Ph = 0$$

Multiplying both sides by (*E/Al*)

$$\frac{\sigma^2}{2} - \sigma\,(\frac{P}{A}) - \frac{PEh}{Al} = 0$$

This is a quadratic equation, we know that

$$\sigma = \frac{P}{A} \pm \sqrt{(\frac{P}{A})^2 + (4 \times \frac{1}{2})\,(\frac{PEh}{Al})}$$

$$= \boldsymbol{\frac{P}{A}\,[1 \pm \sqrt{1 + \frac{2AEh}{Pl}}]}$$

Once the stress (σ) is obtained, the corresponding instantaneous deformation (δl) or the strain energy stored may be found out as usual.

Note: When δl is very small as compared to h, then

$$Work\ done = Ph$$

$$\Rightarrow \frac{\sigma^2}{2E}\,Al = Ph$$

$$\Rightarrow \sigma^2 = \frac{2EPh}{Al}$$

$$\therefore \boldsymbol{\sigma} = \sqrt{\boldsymbol{\frac{2EPh}{Al}}}$$

***Example 12**: A 2 m long alloy bar of 1500 mm^2 cross-sectional area hangs vertically and has a collar securely fixed at its lower end. Find the stress induced in the bar, when a weight of 2 kN falls from a height of 100 mm on the collar. Take E = 120 GPa. Also find the strain energy stored in the bar.*

Solution. Given : Length of bar (l) = 2 m = 2×10^3 mm ; Cross-sectional area of bar (A) = 1500 mm^2 ; Weight falling on collar of bar (P) = 2 kN = 2×10^3 N ; Height from which weight falls (h) = 100 mm and modulus of elasticity (E) = 120 GPa = 120×10^3 N/mm^2.

Stress induced in the bar

We know that in this case, extension of the bar will be small and negligible as compared to the height (h) from where the weight falls on the collar (due to small value of weight *i.e.*, 2 kN and a large value of h *i.e.*, 100 mm). Therefore stress induced in the bar

$$\sigma = \sqrt{\frac{2EPh}{A.l}} = \sqrt{\frac{2 \times (120 \times 10^3) \times (2 \times 10^3) \times 100}{1500 \times (2 \times 10^3)}}\ \text{N/mm}^2$$

$$= 126.5\ \text{N/mm}^2 = 126.5\ \text{MPa} \qquad \textbf{Ans.}$$

Strain energy stored in the bar

We also know that volume of the bar,

$$V = l \cdot A = (2 \times 10^3) \times 1500 = 3 \times 10^6 \text{ mm}^3$$

and strain energy stored in the bar,

$$U = \frac{\sigma^2}{2E} \times V = \frac{(126.5)^2}{2 \times (120 \times 10^2)} \times (3 \times 10^6) \text{ N-mm}$$

$$= 200 \times 10^3 \text{ N-mm} = 200 \text{ N-m} \qquad \textbf{Ans.}$$

<u>Example 13</u>: A steel bar 3 m long and 2500 mm^2 in area hangs vertically, which is securely fixed on a collar at its lower end. If a weight of 15 kN falls on the collar from a height of 10 mm, determine the stress developed in the bar. What will be the strain energy stored in the bar? Take E as 200 GPa.

SOLUTION. Given : Length of bar (l) = 3 m = 3×10^3 mm ; Area of bar (A) = 2500 mm^2 ; Weight falling on collar of bar (P) = 15 kN = 15×10^3 N ; Height from which weight falls (h) = 10 mm and modulus of elasticity (E) = 200 GPa = 20×10^3 N/mm^2.

Stress developed in the bar

We know that in this case, extension of the bar will be considerable as compared to the height (h) from where the weight falls on the collar (due to a large value of weight *i.e.*, 15 kN and a small value

of h = 10 mm). Therefore stress developed in the bar,

$$\sigma = \frac{P}{A}\left[1 + \sqrt{1 + \frac{2AEh}{Pl}}\right]$$

$$= \frac{15 \times 10^3}{2500}\left[1 + \sqrt{1 + \frac{2 \times 2500 \times (200 \times 10^3) \times 10}{(15 \times 10^3) \times (3 \times 10^3)}}\right] \text{ N/mm}^2$$

$$= 6\,(1 + 14.9) = 95.4 \text{ N/mm}^2 = 95.4 \text{ MPa} \qquad \textbf{Ans.}$$

Strain energy stored in the bar

We know that volume of the bar,

$$V = l \cdot A = (3 \times 10^3) \times 2500 = 7.5 \times 10^6 \text{ mm}^3$$

and strain energy stored in the bar,

$$U = \frac{\sigma^2}{2E} \times V = \frac{(95{,}4)^2}{2 \times (200 \times 10^3)} \times 7.5 \times 10^6 \text{ N-mm}$$

$$= 170.6 \times 10^3 \text{ N-mm} = 170.6 \text{ N-m} \qquad \textbf{Ans.}$$

Possible Short Type Question With Answers:

1. Define stress and state its S.I unit.

- The internal resistance per unit area offered by the material of the body against external loading is called intensity of stress or simply called as stress.

Or

- The internal resistance which the body offers to meet with the load is called stress.
- **Mathematically,**

$$\boldsymbol{\sigma = \frac{R}{A} = \frac{P}{A}}$$

- Its unit is N/m^2, N/mm^2, KPa, MPa, GPa.

2. State Hooke's Law. (W – 2019)

- Hooke's law states that when a material is loaded within elastic limit, stress is directly proportional to strain.
- Mathematically,

$$stress \text{ a } strain$$

$$\Rightarrow \frac{stress}{strain} = constant\ (E)$$

Where the constant of proportionality ***E*** is called ***Young's modulus or modulus of elasticity***.

3. Define Strain.

- The strain is the deformation produced by stress.
- The ratio of change in dimension to original dimension of a body is called as strain.
- It is denoted by the letter 'e' or 'ε'.
- It is a unit less quantity.
- Strain(e) $= \frac{change\ in\ dimension}{original\ dimension}$

4. Define Poisson's ratio.

- If a body is stressed within its elastic limit, the lateral strain bears a constant ratio to the linear strain this constant is known as Poisson's ratio.
- It is denoted by symbol 'μ' or 1/m.
- It is unit less.
- Mathematically,

$$\mu = \frac{lateral\ strain}{linear\ strain} = \frac{1}{m}$$

The value of 'μ' lies from 0.25 to 0.34 for different materials.

5. Define young's modulus of elasticity.

- It is defined as, "The ratio of stress to strain".
- It is denoted by the letter ***'E'***.

- Mathematically,

$$E = \frac{stress}{strain} = \frac{\sigma}{e}$$

Its unit is same as stress i.e. N/m^2, N/mm^2, KPa, MPa, GPa.

6. What is meant by modulus of rigidity? (W – 2019)

- It is defined as, "The ratio of shear stress to shear stain".
- It is denoted by letter ***C, N* or *G***.
- Mathematically,

$$C = \frac{shear\ stress}{shear\ strain} = \frac{r}{e_s}$$

- Its unit is N/m^2, N/mm^2, KPa, MPa, GPa.

7. Define temperature stress. (W - 2020)

- It is defined as the stress produced due to prevention of elongation or contraction of a bar in order to increase or decrease of temperature.

8. What is the difference between stress and strain? (W – 2019, 2020)

- The main difference between stress and strain is that stress measures the deforming force per unit area of the object, whereas strain measures the relative change in length caused by a deforming force.
- Stress is measured in Pascal (Pa) but strain has no unit, it is simply a ratio.

9. Define strain energy.

- When an elastic body is loaded within elastic limit, it deforms and some work is done which is stored within the body in the form of internal energy. This stored energy in the deformed body is known as strain energy or potential energy of deformation and is denoted by ***'U'***.

10. What is resilience? (W – 2019, 2020)

- It is the ability of a material to regain its original shape on removal of the applied load. It is defined as the strain energy per unit volume in simple tension or compression and is equal to $\boldsymbol{\sigma^2/2E}$. It is also referred as strain energy density and is denoted by ***'u'***.

Possible Long Type Questions:

1. A steel bar 2 m long and 150 mm^2 in cross section is subjected to an axial pull of 15 KN. Find the elongation of the bar. Take E = 200GPa. (W – 2019)

Hints: Use the formula, $\delta l = \frac{Pl}{AE}$ and refer example – 1

2. Derive the relationship between Young's Modulus of Elasticity & Bulk Modulus. (W – 2019)

Hints: Refer page no – 10

3. A rectangular body 400 mm long, 100 mm wide & 50 mm thick is subjected to a shear stress of 60 MPa. Determine the strain energy stored in the body. Take modulus of rigidity = 80 N/mm^2. (W – 2019)

Hints: Use the formula, $\boldsymbol{U = \frac{r^2}{2C} \times V}$

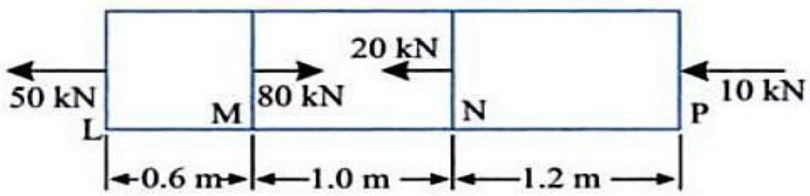
4. A rod 150 cm long and of diameter 2.0 cm is subjected to an axial pull of 20 KN. If the modulus of elasticity of the material of the rod is 2 × 10^5 N/mm^2, determine (i) the stress (ii) the strain (iii) the elongation of the rod. (W – 2020)

Hints: Use the formula, $\boldsymbol{= \frac{P}{A}, e = \frac{\sigma}{E}, \delta l = \frac{Pl}{AE} \quad or \quad \delta l = e \times l}$

5. Derive the relationship between Young's Modulus of Elasticity & Modulus of Rigidity. (W – 2020)

Hints: Refer page no – 09

6. A brass bar having cross sectional area of 1000 mm^2 is subjected to axial forces shown in the figure. Find the total elongation of the bar. Modulus of elasticity of brass is 100 GN/m^2. (W – 2020)



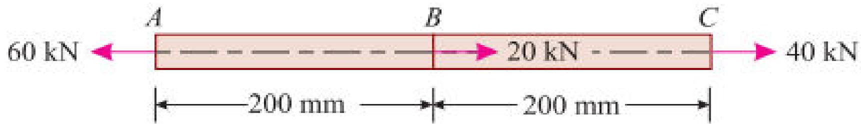
Hints: Use the formula,

$$\delta l = \frac{P_1 l_1}{AE} + \frac{P_2 l_2}{AE} + \frac{P_3 l_3}{AE} \dots\dots\dots$$

7. A reinforced short concrete column 250 mm × 250 mm in section is reinforced with 8 steel bars. The total area of steel bars is 2500 mm^2. The column carries a load of 390 KN. If the modulus of elasticity for steel is 15 times that of concrete, find the stresses in concrete and steel. (W – 2020)

Hints: Refer page no – 15 (Example – 7)

8. A steel bar ABC of 400 mm length and 20 mm diameter is subjected to a point loads as shown in Figure. Determine the total change in length of the bar. Take E = 200 GPa.



[Ans. 0.32 mm]

9. A reinforced concrete column of 300 mm diameter contains 4 bars of 22 mm diameter. Find the total load, the column can carry, if the stresses in steel and concrete are 50 MPa and 3 MPa respectively.

[Ans. 283.5 KN]

10. An aluminium rod of 20 mm diameter is completely enclosed in a steel tube of 30 mm external diameter and both the ends of the assembly are rigidly connected. If the composite bar is heated through 50°C, find the stresses developed in the aluminium rod and steel tube. Take:

Modulus of elasticity for steel = 200 GPa

Modulus of elasticity for aluminium = 80 GPa

Coefficient of expansion for steel = 12×10^{-6}/°C

Coefficient of expansion for aluminium = 18×10^{-6}/°C

[Ans. 14.5 MPa (Comp.); 18.1 MPa (Tension)]

11. A mild steel rod 1 m long and 20 mm diameter is subjected to an axial pull of 62.5 kN. What is the elongation of the rod, when the load is applied (i) gradually and (ii) suddenly? Take E as 200 GPa.

[Ans. 1mm; 2mm]

CHAPTER 02

THIN CYLINDER AND SPHERICAL SHELL UNDER INTERNAL PRESSURE

LEARNING OBJECTIVES:

Definition of hoop and longitudinal stress, strain
Derivation of hoop stress, longitudinal stress, hoop strain, Longitudinal strain and volumetric strain
Computation of the change in length, diameter and volume
Simple problems on above

Introduction:

- Thin pressure vessels or shells are used to carry fluid. The thickness of the wall is small as compared to the diameter. These are made by rolling the sheet metal and joining the ends by riveting or welding. Example: boiler shell, gas cylinder, water tank, pipe line carrying fluid under pressure.
- A pressure vessel is a thin one if the ratio of its internal diameter to wall thickness ≥ 20, i.e.

$$\frac{d}{t} \geq 20 \quad or \quad \frac{t}{d} \leq \frac{1}{20}$$

  Otherwise it is a thick shell.
- Shape of shells may be cylindrical, spherical, cylindrical shell with hemispherical ends.

Definition Of Hoop And Longitudinal Stress, Strain

Stresses in a Thin Cylindrical Shell:

- Whenever a cylindrical shell is subjected to an internal pressure, its walls are subjected to tensile stresses. The walls of the cylindrical shell will be subjected to the following types of stresses.
  1. Circumferential stress and
  2. Longitudinal stress
  3. Radial stress

Hoop stress or circumferential stress:

- It is a tensile stress acting along the circumference of the cylinder.
- It is denotes by (σ_c).

Hoop strain:

- Circumference depends on the diameter of the shell. So it is the ratio of change in diameter to the original diameter of the shell.
- Mathematically,

$$e_c = \frac{\delta d}{d}$$

Longitudinal stress:

- It is a tensile stress acting along the length of the cylinder. It develops only if cylinder has closed ends.
- It is denoted by (σ_l).

Longitudinal strain:

- It is the ratio of change in length to the original length of the shell.
- Mathematically,

$$e_l = \frac{\delta l}{l}$$

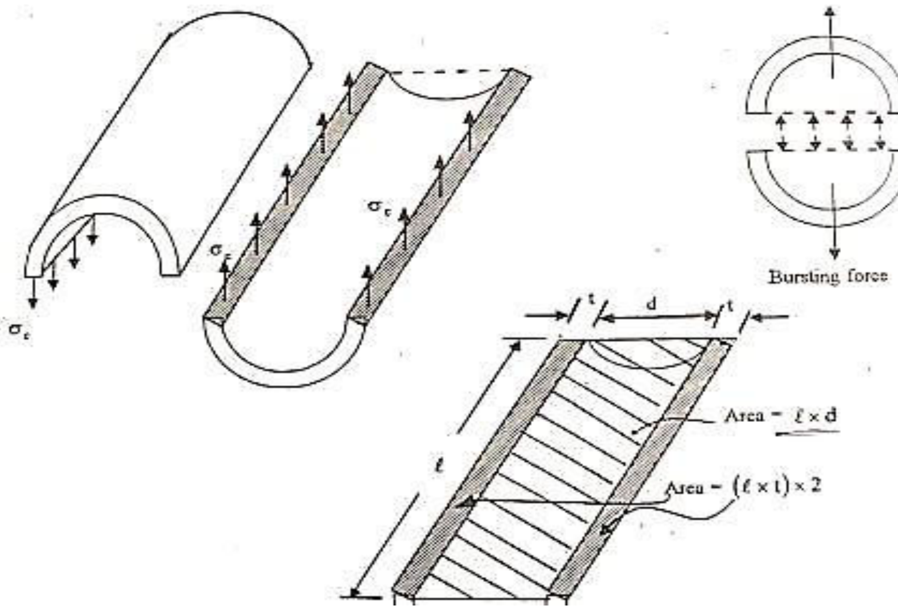
Radial stress:

- It is a compressive stress acting along the radius of the cylinder. It is small and neglected.
- It is denoted by (σ_r).

Derivation of hoop stress, longitudinal stress, hoop strain, longitudinal strain and volumetric strain

Derivation of Hoop Stress in a Cylindrical Shell:

Consider a thin cylindrical shell subjected to an internal pressure.



Let P = Internal fluid pressure inside the cylinder

σ_c = Circumferential or Hoop stress

d = Internal diameter of the cylinder

t = Thickness of the cylinder

l = Length of the cylinder

Bursting force = internal fluid pressure × area

= $P \times l \times d$

Resisting force = circumferential stress × area on which it acts

= $\sigma_c \times (2 \times l \times t)$

Considering the equilibrium of a half of the cylinder

Resisting force = bursting force

$\Rightarrow \sigma_c \times (2 \times l \times t) = P \times l \times d$

$\Rightarrow \boldsymbol{\sigma_c = \frac{Pd}{2t}}$

Note: if $\boldsymbol{\eta_l}$ is the efficiency of longitudinal joints of the shell, then hoop stress

$$\sigma_c = \frac{Pd}{2t\eta_l}$$

Derivation of Longitudinal Stress in a Cylindrical Shell:

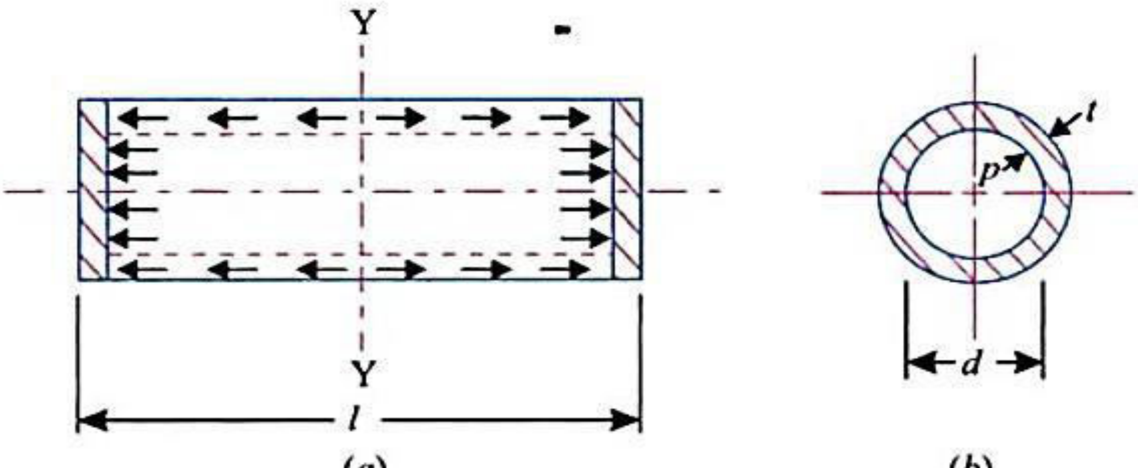
Let P = Internal fluid pressure

σ_l = Circumferential or Hoop stress

d = Internal diameter of the cylinder

t = Thickness of the cylinder

l = Length of the cylinder



(a) (b)

Bursting force = internal fluid pressure × area

$$= P \times \frac{\pi}{4} \times d^2$$

Resisting force = longitudinal stress × area on which it acts

$$= \sigma_l \times \pi \times dt$$

For equilibrium,

Resisting force = bursting force

$$\Rightarrow \sigma_l \times \pi \times dt = P \times \frac{\pi}{4} \times d^2$$

$$\Rightarrow \boldsymbol{\sigma_l = \frac{Pd}{4t}}$$

Note 1: If $\boldsymbol{\eta_c}$ is the efficiency of circumferential joints of the shell, then longitudinal stress

$$\sigma_l = \frac{Pd}{2t\eta_c}$$

Note 2: In case of thin cylinder subjected to internal fluid pressure, the hoop stress developed is twice that of the longitudinal stress.

$$\boldsymbol{\sigma_c = 2\sigma_l}$$

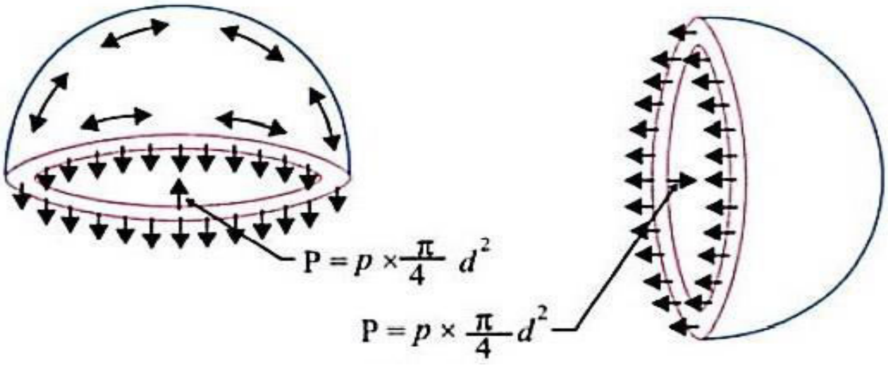
Derivation of Hoop Stress in a Spherical Shell:

Let P = Internal fluid pressure inside the shell

σ_c = Circumferential or Hoop stress

d = Internal diameter of the shell

t = Thickness of the shell



Bursting force = internal fluid pressure × area

$$= P \times \frac{\pi}{4} \times d^2$$

Resisting force = circumferential stress × area on which it acts

$$= \sigma_c \times \pi \times dt$$

For equilibrium

Resisting force = bursting force

$$\Rightarrow \sigma_c \times \pi \times dt = P \times \frac{\pi}{4} \times d^2$$

$$\Rightarrow \boldsymbol{\sigma_c = \frac{Pd}{4t}}$$

Note 1: If $\boldsymbol{\eta_c}$ is the efficiency of the circumferential joints of the spherical shell, then hoop stress,

$$\sigma_c = \frac{Pd}{4t\eta_c}$$

Note 2: There is no longitudinal stress in spherical shells so σ_c (Hoop stress) is the only stress induced in the spherical shells.

Hoop Strain, Longitudinal Strain and Volumetric Strain:

Let l = Length of the shell
d = Diameter of the shell
t = Thickness of the shell
P = Intensity of pressure
μ = 1/m = poisson's ratio
σ_c = Hoop stress or circumferential stress
σ_l = Longitudinal stress

Direct strain (e_c) due to $\sigma_c = \frac{\sigma_c}{E}$
Direct strain (e_l) due to $\sigma_l = \frac{\sigma_l}{E}$

For cylindrical shells:

- **Net Circumferential strain or hoop strain,**

$$e_c = \frac{\sigma_c}{E} - \frac{\sigma_l}{mE}$$
$$= \frac{Pd}{2tE} - \frac{1}{m} \times \frac{Pd}{4tE}$$
$$= \frac{Pd}{2tE}\left(1 - \frac{1}{2m}\right)$$

- **Net Longitudinal strain,**

$$e_l = \frac{\sigma_l}{E} - \frac{\sigma_c}{mE}$$
$$= \frac{Pd}{4tE} - \frac{1}{m} \times \frac{Pd}{2tE}$$
$$= \frac{Pd}{4tE}\left(1 - \frac{2}{m}\right)$$

- **Volumetric strain,**

$$e_v = Algebric\ sum\ of\ net\ strains\ in\ all\ axes$$
$$= net\ longitudinal\ strain + 2 \times net\ hoop\ strain$$
$$= e_l + 2e_c$$

For spherical shells:

- **Circumferential strain or hoop strain,**

$$e_c = \frac{\sigma_c}{E} - \frac{\sigma_c}{mE}$$
$$= \frac{\sigma_c}{E}(1 - \frac{1}{m})$$
$$= \frac{Pd}{4tE}(1 - \frac{1}{m})$$

- **Volumetric strain,**

$$e_v = Algebric\ sum\ of\ strains\ in\ all\ the\ three\ axis$$
$$= e_c + e_c + e_c = 3e_c$$

Computation of the change in length, diameter and volume

For cylindrical shells:

- **Change in length:**
  Change in length depends upon longitudinal strain.

$$e_l = \frac{\delta l}{l}$$
$$\Rightarrow \delta l = e_l \times l$$
$$= \frac{Pd}{4tE}(1 - \frac{2}{m}) \times l$$
$$= \frac{Pdl}{4tE}(1 - \frac{2}{m})$$

- **Change in diameter:**
  Change in diameter depends upon circumferential or hoop strain.

$$e_c = \frac{\delta d}{d}$$
$$\Rightarrow \delta d = e_c \times d$$
$$= \frac{Pd}{2tE}(1 - \frac{1}{2m}) \times d$$
$$= \frac{Pd^2}{2tE}(1 - \frac{1}{2m})$$

- **Change in volume:**
  Change in volume depends upon volumetric strain.

$$e_v = \frac{\delta V}{V}$$
$$\Rightarrow \delta V = e_v \times V$$
$$= (e_l + 2e_c) \times V$$
$$= [\frac{Pd}{4tE}(1 - \frac{2}{m}) + 2 \times \frac{Pd}{2tE}(1 - \frac{1}{2m})] \times V$$
$$= \frac{Pd}{2tE}[(\frac{1}{2} - \frac{1}{m}) + (2 - \frac{1}{m})] \times V$$
$$= \frac{PdV}{2tE}(\frac{5}{2} - \frac{2}{m})$$

$[Where, V = \frac{\pi}{4} d^2 l\ ]$

For spherical shells:

- **Change in diameter:**
  Change in diameter depends upon circumferential or hoop strain.

$$e_c = \frac{\delta d}{d}$$
$$\Rightarrow \delta d = e_c \times d$$
$$= \frac{Pd}{4tE}(1 - \frac{1}{m}) \times d$$
$$= \frac{Pd^2}{4tE}(1 - \frac{1}{m})$$

- **Change in volume:**
  Change in volume depends upon volumetric strain.

$$e_v = \frac{\delta V}{V}$$
$$\Rightarrow \delta V = e_v \times V$$
$$= 3e_c \times V$$
$$= \frac{3Pd}{4tE}(1 - \frac{1}{m}) \times V$$

$[Where, V = \frac{4}{3}\pi r^3 \;or\; \frac{\pi d^3}{6}]$

Simple problems on above:

Example 1: A stream boiler of 800 mm diameter is made up of 10 mm thick plates. If the boiler is subjected to an internal pressure of 2.5 MPa, find the circumferential and longitudinal stresses induced in the boiler plates.

Solution. Given : Diameter of boiler (d) = 800 mm ; Thickness of plates (t) = 10 mm and internal pressure (p) = 2.5 MPa = 2.5 N/mm^2.

Circumferential stress induced in the boiler plates

We know that circumferential stress induced in the boiler plates,

$$\sigma_c = \frac{pd}{2t} = \frac{2.5 \times 800}{2 \times 10} = 100 \text{ N/mm}^2 = \mathbf{100 \text{ MPa}} \quad \textbf{Ans.}$$

Longitudinal stress induced in the boiler plates

We also know that longitudinal stress induced in the boiler plates,

$$\sigma_l = \frac{pd}{4t} = \frac{2.5 \times 800}{4 \times 10} = 50 \text{ N/mm}^2 = \mathbf{50 \text{ MPa}} \quad \textbf{Ans.}$$

Example 2: A cylindrical shell of 1.3 m diameter is made up of 18 mm thick plates. Find the circumferential and longitudinal stress in the plates, if the boiler is subjected to an internal pressure of 2.4 MPa. Take efficiency of the joints as 70%.

SOLUTION. Given: Diameter of shell (d) = 1.3 m = 1.3×10^3 mm ; Thickness of plates (t) = 18 mm; Internal pressure (p) = 2.4 MPa = 2.4 N/mm^2 and efficiency (η) = 70% = 0.7.

Circumferential stress

We know that circumferential stress,

$$\sigma_c = \frac{pd}{2t\eta} = \frac{2.4 \times (1.3 \times 10^3)}{2 \times 18 \times 0.7} = 124 \text{ N/mm}^2 = \mathbf{124 \text{ MPa}} \qquad \textbf{Ans.}$$

Longitudinal stress

We also know that longitudinal stress,

$$\sigma_l = \frac{pd}{4t\eta} = \frac{2.4 \times (1.3 \times 10^3)}{4 \times 18 \times 0.7} = 62 \text{ N/mm}^2 = \mathbf{62 \text{ MPa}} \qquad \textbf{Ans.}$$

Example 3: A cylindrical thin drum 800 mm in diameter and 4 m long is made of 10 mm thick plates. If the drum is subjected to an internal pressure of 2.5 MPa, determine its changes in diameter and length. Take E as 200 GPa and Poisson's ratio as 0.25.

SOLUTION. Given: Diameter of drum (d) = 800 mm ; Length of drum (l) = 4 m = 4×10^3 mm ; Thickness of plates (t) = 10 mm ; Internal pressure (p) = 2.5 MPa = 2.5 N/mm^2 ; Modulus of elasticity (E) = 200 GPa = 200×10^3 N/mm^2 and poisson's ratio $\left(\frac{1}{m}\right)$ = 0.25.

Change in diameter

We know that change in diameter,

$$\delta d = \frac{pd^2}{2tE}\left(1 - \frac{1}{2m}\right) = \frac{2.5 \times (800)^2}{2 \times 10 \times (200 \times 10^3)}\left(1 - \frac{0.25}{2}\right) \text{mm}$$

$$= \mathbf{0.35 \text{ mm}} \qquad \textbf{Ans.}$$

Change in length

We also know that change in length,

$$\delta l = \frac{pdl}{2tE}\left(\frac{1}{2} - \frac{1}{m}\right) = \frac{2.5 \times 800 \times (4 \times 10^3)}{2 \times 10 \times (200 \times 10^3)}\left(\frac{1}{2} - 0.25\right) \text{mm}$$

$$= \mathbf{0.5 \text{ mm}} \qquad \textbf{Ans.}$$

Example 4: A spherical gas vessel of 1.2 m diameter is subjected to a pressure of 1.8 MPa. Determine the stress induced in the vessel plate, if its thickness is 5 mm.

SOLUTION. Given: Diameter of vessel (d) = 1.2 m = 1.2×10^3 mm ; Internal pressure (p) = 1.8 MPa = 1.8 N/mm^2 and thickness of plates (t) = 5 mm.

We know that stress in the vessel plates,

$$\sigma = \frac{pd}{4t} = \frac{1.8 \times (1.2 \times 10^3)}{4 \times 5} = 108 \text{ N/mm}^2 = \mathbf{108 \text{ MPa}} \qquad \textbf{Ans.}$$

Possible Short Type Question With Answers:

1. Define hoop stress. (W – 2019, 2020)

- It is a tensile stress acting along the circumference of the cylinder.
- It is denotes by ($\boldsymbol{\sigma_c}$).

2. Define longitudinal stress. (W - 2020)

- It is a tensile stress acting along the length of the cylinder. It develops only if cylinder has closed ends.
- It is denoted by (σ_l).

Possible Long Type Questions:

1. A cylindrical shell 2 m long and 1 m internal diameter is made up of 20 mm thick plates. Find the circumferential stress and longitudinal stress in the shell material, if it is subjected to an internal pressure of 5 MPa. **(W - 2019)**

Hints: Refer example 01 (page no 06)

2. Derive the expression for hoop stress and longitudinal stress in case of a thin cylindrical shell. **(W - 2020)**

Hints: Refer article no 2.2 (page no 2 & 3)

3. A steam boiler of 1.25 m in diameter is subjected to an internal pressure of 1.6 MPa. If the steam boiler is made up of 20 mm thick plates, calculate the circumferential and longitudinal stresses. Take efficiency of the circumferential and longitudinal joints as 75% and 60% respectively.

Hints: Refer example 02 (page no 07)

CHAPTER NO -03

PROPERTIES PROCESSES OF PERFECT GAS

INTRODUCTION: A perfect gas or an ideal gas may be defined as a state of a substance, whose evaporation from its liquid state is complete and strictly obeys all the gas laws under all conditions of pressure and temperature.

LAWS OF PERFECT GAS:

Boyle's Law: This law was formulated by Robert Boyle in 1662. It states, ***"The absolute pressure of a given mass of a perfect gas varies inversely as its volume, when the temperature remains constant".***

Mathematically, $\boldsymbol{p \propto \frac{1}{v} \quad or \quad pv = constant}$

$$or\ p_1v_1 = p_2v_2 = p_3v_3 = \cdots = constant$$

Charles's Law: This law was formulated by a Frenchman Jacques A.C. Charles in about 1787. It states, ***"The volume of a given mass of a perfect varies directly as its absolute temperature, when the absolute pressure remains constant".***

Mathematically, $\boldsymbol{v \propto T \quad or \quad \frac{v}{T} = constant}$

$$or\ \frac{v_1}{T_1} = \frac{v_2}{T_2} = \frac{v_3}{T_3} = \cdots = constant$$

Gay-Lussac Law: This law states, ***"The absolute pressure of a given mass of a perfect gas varies directly as its absolute temperature when the volume remains constant".***

Mathematically, $\boldsymbol{p \propto T \quad or \quad \frac{p}{T} = constant}$

$$or\ \frac{p_1}{T_1} = \frac{p_2}{T_2} = \frac{p_3}{T_3} = \cdots = constant$$

Avogadro's Law: This law states, ***"Equal volumes of all gases, at the same temperature and pressure, contain equal no of molecules".***

- Avogadro's experiments shows that the average volume for 1 Kg-mole of any perfect gas is 22.413 m^3 at standard atmospheric pressure (1.01325 bar) and 0^0C NTP condition.

General Gas Equation: The Boyle's law and Charles's law are combined together, which give us a general gas equation.

According to Boyle's law

⇨ $p \propto \frac{1}{v}$ or $v \propto \frac{1}{p}$

According to Charles's law

⇨ $v \propto T$

It is thus obvious that

⇨ $v \propto \frac{T}{p}$

⇨ $pv \propto T$

⇨ $pv = CT$

⇨ $\boldsymbol{\frac{pv}{T} = C}$

Where ***C*** is a constant, whose value depends upon the mass and properties of the gas.

⇨ $\frac{p1v1}{T_1} = \frac{p_2v_2}{T_2} = \frac{p_3v_3}{T_3} = \cdots = constant$

Characteristic Equation of Gas: It is a modified form of general gas equation. If the volume (***v***) in the general gas equation is taken as that of 1kg of gas (known as its specific volume, and denoted by $\boldsymbol{v_s}$), then the constant ***C*** (in the general gas equation) is represented by another constant ***R*** (in the characteristic equation of gas).

Thus the general gas equation may be rewritten as:

$$\boldsymbol{pv_s = RT}$$

Where, ***R*** is known as ***characteristic gas constant*** or simply ***gas constant***.

For any mass m kg of a gas, the characteristic gas equation becomes:

$$mpv_s = mRT$$
$$\textbf{or} \quad \boldsymbol{pv = mRT} \qquad \ldots.(mv_s = v)$$

- In S.I. units, the value of ***R*** for atmospheric air is taken as ***0.287 kJ/kg K or 287 J/kg K***.

<u>Universal Gas Constant</u>: The universal gas constant or molar constant of a gas is the product of the gas constant and the molecular mass of the gas. It is generally denoted by $\boldsymbol{R_u}$.

Mathematically,

$$\boldsymbol{R_u = MR}$$

Where, M = Molecular mass of the gas expressed in kg-mole
R = Gas constant

- The value of $\boldsymbol{R_u}$ is same for all gases.
- In S.I. units, the value of $\boldsymbol{R_u}$ is taken as 8314 ***J/kg-mol K or 8.314 kJ/kg-mol K***

<u>Dalton's Law of Partial Pressure</u>: When two or more gases which don't react chemically with one another are enclosed in a vessel, then the total pressure exerted by the mixture of gases will be equal to the sum of the partial pressures which each gas would exert if present alone in that space.

Mathematically,

$$\boldsymbol{p = p_a + p_b + p_c \ldots \ldots \ldots \ldots \ldots}$$

Where, p = Total pressure of the mixture of gases
p_a, p_b, p_c = Partial pressures of each gas

<u>Example 3.1</u>*: *A gas occupies a volume of 0.1 m^3 of a temperature of 20^0 C and a pressure of 1.5 bar. Find the final temperature of the gas, if it is compressed to a pressure of 7.5 bar and occupies a volume of 0.04 m^3.

<u>Solution</u>: Data given: $v_1 = 0.1\ m^3$, $T_1 = 20^0\ C = 20 + 273 = 293\ K$, $p_1 = 1.5\ bar = 1.5 \times 10^5\ N/m^2$, $p_2 = 7.5\ bar = 7.5 \times 10^5\ N/m^2$, $v_2 = 0.04\ m^3$, $T_2 = ?$

According to general gas equation,

We know that, $\frac{p_1v_1}{T_1} = \frac{p_2v_2}{T_2}$

$$\Rightarrow T_2 = \frac{p_2v_2T_1}{p_1v_1}$$

$$\Rightarrow T_2 = \frac{7.5 \times 10^5 \times 0.04 \times 293}{1.5 \times 10^5 \times 0.1}$$

$$\Rightarrow T_2 = 586\ K\ or\ 313^0C \qquad \text{Ans.}$$

<u>Example 3.2</u>*: *A vessel of capacity 3 m^3 contains air at a pressure of 1.5 bar and a temperature of 25^0 C. Additional air is now pumped into the system until the pressure rises to 30 bar and temperature rises to 60^0 C. Determine the mass of air pumped in and express the quantity as a volume at a pressure of 1.02 bar and a temperature of 20^0 C. If the vessel is allowed to cool until the temperature is again 25^0 C, calculate the pressure in the vessel.

<u>Solution</u>: Data given: $v_1 = 3\ m^3$, $p_1 = 1.5\ bar = 1.5 \times 10^5\ N/m^2$, $T_1 = 25^0\ C = 25 + 273 = 298\ K$, $p_2 = 30\ bar = 30 \times 10^5\ N/m^2$, $T_2 = 60^0\ C = 60 + 273 = 333\ K$, $p_3 = 1.02\ bar = 1.02 \times 10^5\ N/m^2$, $T_3 = 20^0\ C = 20 + 273 = 293\ K$

Mass of air pumped in

We know that, $p_1v_1 = m_1RT_1$

$$\Rightarrow m_1 = \frac{p_1v_1}{RT_1} = \frac{1.5 \times 10^5 \times 3}{287 \times 298} = 5.26\ kg \qquad \text{.......(Taking } R \text{ for air} = 287\ J/kg\ K)$$

Similarly, $p_2 v_2 = m_2 R T_2$

$$\Rightarrow \quad m_2 = \frac{p_2 v_2}{R T_2} = \frac{30 \times 10^5 \times 3}{287 \times 333} = 94.17\ kg \qquad \text{.......}(\because v_2 = v_1)$$

$\therefore$ Mass of air pumped in, $m = m_2 - m_1 = 94.17 - 5.26 = 88.91\ kg$ Ans.

Volume of air pumped in at a pressure of 1.02 bar and temperature of 20⁰ C

Let v_3 = volume of air pumped in.

We know that, $p_3 v_3 = mRT_3$

$$\Rightarrow \quad v_3 = \frac{mRT_3}{p_3} = \frac{88.91 \times 287 \times 293}{1.02 \times 10^5} = 73.29\ m^3 \qquad \text{Ans.}$$

Pressure in the vessel after cooling

Let p_4 = pressure in the vessel after cooling

We know that the temperature after cooling, $T_4 = T_1 = 25^0\ C = 298\ K$

Since the cooling is at constant volume, therefore

$$\frac{p_2}{T_2} = \frac{p_4}{T_4}$$

$$\Rightarrow \quad p_4 = \frac{T_4 p_2}{T_2} = \frac{298 \times 30 \times 10^5}{333} = 26.8 \times 10^5\ N/m^2$$

$$\Rightarrow \quad p_4 = 26.8\ bar \qquad \text{Ans.}$$

<u>SPECIFIC HEATS OF A GAS</u>: The specific heat of a substance may be broadly defined as the amountof heat required to raise the temperature of its unit mass through one degree.

Mathematically,

$$\boldsymbol{C = \frac{Q}{m \times \Delta t}}$$

Where, C = Specific heat of a substance
Q = Amount of heat transfer
m = Mass of the substance
Δt = Rise in temperature

- S.I. unit, ***J/kg K or kJ/kg K***
- All the liquids and solids have one specific heat only. But a gas can have any number of specific heats (lying between zero and infinity) depending upon the conditions, under which it is heated.

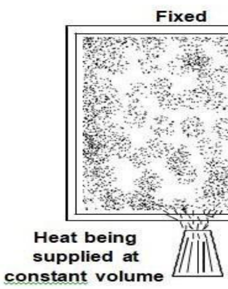
The following two types of specific heats of a gas are important from the subject point of view.

- Specific heat at constant volume
- Specific heat at constant pressure

<u>Specific Heat at Constant Volume</u>: It is the amount of heat required to raise the temperature of a unit mass of gas through one degree when it is heated at a constant volume.

- It is generally denoted by ***'c_v'.***

Consider a gas contained in a container with a fixed lid as shown in the figure. Now, if this gas is heated, it will increase the temperature and pressure of the gas in the container. Since the lid of the container is fixed, therefore the volume of gas remains unchanged.



Let m = Mass of the gas
T_1 = Initial temperature of the gas
T_2 = Final temperature of the gas

∴ Total heat supplied to the gas at constant volume,

$$Q_{1-2} = Mass \times Specific\ heat\ at\ constant\ volume \times Rise\ in\ temperature$$

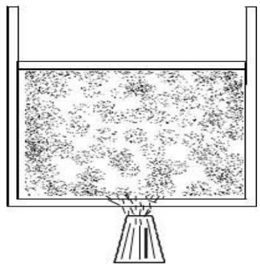
$$\Rightarrow \quad Q_{1-2} = mc_v(T_2 - T_1)$$

It may be noted that whenever a gas is heated at constant volume, no work is done by the gas. The whole heat energy is utilised in increasing the temperature and pressure of the gas. In other words, all the amount of heat supplied remains within the gas, and represents the increase in internal energy of the gas.

Specific Heat at Constant Pressure: It is the amount of heat required to raise the temperature of a unit mass of a gas through one degree, when it is heated at constant pressure.

- It is generally denoted by 'c_p'.

Consider a gas contained in a container with a movable lid as shown in figure. Now if this gas is heated, it will increase the temperature and pressure of the gas in container. Since the lid of the container is movable, therefore it will move upwards in order to counter balance the tendency for pressure to rise.



Heat being supplied at constant pressure

Let m = Mass of the gas
T_1 = Initial temperature of the gas
T_2 = Final temperature of the gas
v_1 = Initial volume of the gas
v_2 = Final volume of the gas

∴ Total heat supplied to the gas at constant pressure,

$$Q_{1-2} = Mass \times Specific\ heat\ at\ constant\ pressure \times Rise\ in\ temperature$$

$$\Rightarrow \quad Q_{1-2} = mc_p(T_2 - T_1)$$

Whenever a gas is heated at a constant pressure, the heat supplied to the gas is utilised for the following two purposes.

1. To raise the temperature of the gas. This heat remains within the gas and represents the increase in internal energy.
   Mathematically, increase in internal energy,
   $$dU = mc_v(T_2 - T_1)$$
2. To do some external work during expansion.
   Mathematically, work done by the gas,
   $$W_{1-2} = p(v_2 - v_1) = mR(T_2 - T_1)$$

RELATION BETWEEN (Cp & Cv):

Consider a gas enclosed in a container and being heated, at a constant pressure, from the initial state 1 to the final state 2.

Let m = Mass of the gas
T_1 = Initial temperature of the gas
T_2 = Final temperature of the gas
v_1 = Initial volume of the gas

v_2 = Final volume of the gas
c_p = Specific heat at constant pressure
c_v = Specific heat at constant volume
p = Constant pressure of the gas

We know that the heat supplied to the gas at constant pressure,

$$Q_{1-2} = mc_p(T_2 - T_1)$$

A part of this heat is utilised in doing the external work, and the rest remains within the gas and is used in increasing the internal energy of the gas

So Heat utilised for external work, $W_{1-2} = p(v_2 - v_1) = mR(T_2 - T_1)$
And increase in internal energy, $dU = mc_v(T_2 - T_1)$

We know that, $Q_{1-2} = dU + W_{1-2}$

⇨ $mc_p(T_2 - T_1) = mc_v(T_2 - T_1) + mR(T_2 - T_1)$

⇨ $c_p = c_v + R$

⇨ $\boldsymbol{c_p - c_v = R}$

⇨ $c_v\left(\frac{c_p}{c_v} - 1\right) = R$

⇨ $c_v(\gamma - 1) = R$ $\qquad\qquad (where\ \frac{c_p}{c_v} = \gamma)$

⇨ $\boldsymbol{c_v = \frac{R}{\gamma-1}}$

<u>ENTHALPY OF A GAS</u>: Enthalpy, a property of a thermodynamic system, is the sum of the system's internal energy (*U*) and the product its pressure (*P*) and volume (*V*).

Mathematically,

$$\boldsymbol{H = U + pv}$$

- **Unit:** ***J, kj***
- The enthalpy of unit mass system is known as specific enthalpy and is denoted by **'*h*'**
- **Unit:** ***kj/kg***
- **Mathematically,**

$$\boldsymbol{h = u + pv_s}$$

<u>Example 3.3</u>: A closed vessel contains 2 kg of carbon dioxide at temperature 20^0 C and pressure 0.7 bar. Heat is supplied to the vessel till the gas acquires a pressure of 1.4 bar. Calculate: 1. Final temperature, 2. Work done on or by the gas, 3. Heat added, and 4. Change in internal energy. Take specific heat of the gas at constant volume as 0.657 kj/kg K.

<u>Solution</u>: Data given: $m = 2\ kg$, $T_1 = 20^0\ C = 20 + 273 = 293\ K$, $p_1 = 0.7\ bar = 0.7 \times 10^5\ N/m^2$, $p_2 = 1.4\ bar = 1.4 \times 10^5\ N/m^2$, $C_V = 0.657\ kj/kg\ K$

1. Final temperature

According to Gay-Lussac Law

$$\frac{p_1}{T_1} = \frac{p_2}{T_2}$$

⇨ $T_2 = \frac{p_2 T_1}{p_1} = \frac{1.4 \times 293}{0.7} = 586\ K\ or\ 313^0\ C$

2. Work done on or by the gas

Since there is no change in volume, therefore work done (***W_{1-2}***) on or by the gas is zero.

3. Heat added

We know that, $\quad Q_{1-2} = mc_v(T_2 - T_1) = 2 \times 0.657\ (586 - 293) = 385\ kj$ $\qquad$ Ans.

4. Change in internal energy

We know that, $\quad Q_{1-2} = W_{1-2} + dU$

⇨ $dU = Q_{1-2} = 385\ kj$ $\qquad$ Ans.

***Example 3.4**: One kg of a perfect gas occupies a volume of 0.85 m^3 at 15^0 C and at a constant pressure of 1 bar. The gas is first heated at constant volume, and then at a constant pressure. Find the specific heat at constant volume and constant pressure of the gas. Take $\gamma = 1.4$.*

Solution: Data given: $m = 1kg$, $v = 0.85\ m^3$, $T = 15^0\ C = 15 + 273 = 288\ K$, $p = 1\ bar = 1 \times 10^5\ N/m^2$, $\gamma = c_p/c_v = 1.4$

Specific heat of gas at constant volume

We know that, $pv = mRT$

$$\Rightarrow \quad R = \frac{pv}{mT} = \frac{1 \times 10^5 \times 0.85}{1 \times 288} = 295\ J/kgK = 0.295\ kj/kgK$$

We also know that, $c_v = \frac{R}{\gamma - 1} = \frac{0.295}{1.4 - 1} = 0.7375\ kj/kgK$ Ans.

Specific heat of gas at constant pressure

We know that, $c_p = 1.4 c_v = 1.4 \times 0.7375 = 1.0325\ kj/kgK$ Ans.

***Example 3.5**: The volume of air at a pressure of 5 bar and 47^0 C is 0.5 m^3. Calculate the mass of air, if the specific heats at constant pressure and volume are 1 kj/kg K and 0.72 kj/kg K respectively.*

Solution: Data given: $p = 5\ bar = 5 \times 10^2\ KN/m^2$, $T = 47^0\ C = 47 + 273 = 320\ K$, $v = 0.5\ m^2$, $c_p = 1\ kj/kg\ K$, $c_v = 0.72\ kj/kg\ K$

Mass of air

We know that, $c_p - c_v = R$

$$\Rightarrow \quad R = 1 - 0.72 = 0.28\ kj/kgK$$

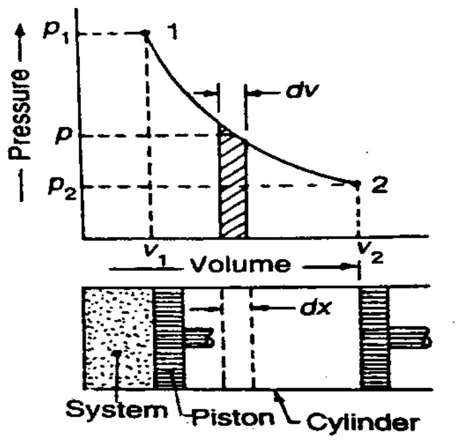
We know that, $pv = mRT$

$$\Rightarrow \quad m = \frac{pv}{RT} = \frac{5 \times 10^2 \times 0.5}{0.28 \times 320} = 2.79\ kg$$ Ans.

Non-flow Processes: The processes occurring in closed systems which do not permit the transfer of mass across their boundaries are known as non-flow processes.

- It may be noted that in a non-flow process, the energy crosses the system boundary in the form heat and work but, there is no mass flow into or out of the system.

WORK DONE DURING A NON-FLOW PROCESS: Consider a system contained in a frictionless piston and cylinder arrangement as shown in the figure below. As the system expands from its initial state 1 to final state 2, some work must be done.



Let at any small section (shown shaded), the pressure (p) of the system is constant. If A is the cross-sectional area of the piston, then force on the piston ($F = p\ A$) causes the piston to move through a distance dx,

Thus work done by the system,

$$\delta W = F\ dx = p\ A\ dx = p\ dv \qquad \ldots\ldots\ldots (\because dv = A\ dx)$$

∴ Work done for non-flow process from state 1 to state 2,

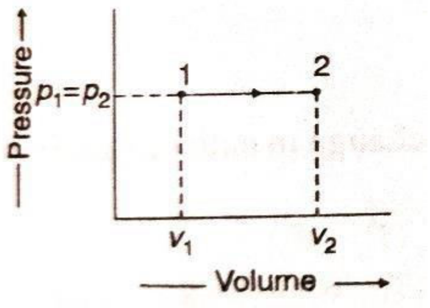
$$W_{1-2} = \int_1^2 \delta W = \int_1^2 p\, dv$$

From above, we see that the work done is given by the area under the *p-v* diagram.

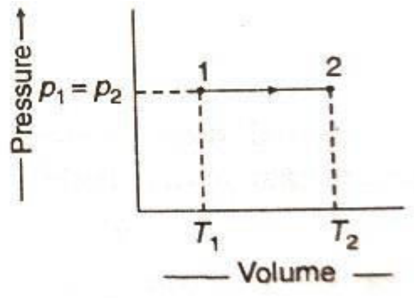
Application of First Law of Thermodynamics to Various Non-flow Processes:

Isobaric Process (Constant Pressure Process): This process is governed by Charles's Law.
Consider *m* kg of a certain gas, contained in a closed container, is being heated at constant pressure from an initial state 1 to a final state 2.

Let p_1, v_1 and T_1 = Pressure, volume and temperature of the gas at the initial state 1.
p_2, v_2 and T_2 = Pressure, volume and temperature of the gas at the final state 2.



(*a*) *p-v* diagram.



(*b*) *p-T* diagram.

Now let us derive the following relations for the reversible isobaric process.

1. Pressure-volume-temperature (p-v-T) relationship

We know that the general gas equation is

$$\frac{p_1 v_1}{T_1} = \frac{p_2 v_2}{T_2}$$

Since the gas is heated at constant pressure, therefore $p_1 = p_2$

$$\frac{v_1}{T_1} = \frac{v_2}{T_2} = \frac{v}{T} = Constant$$

Thus, the isobaric process is governed by Charles's Law.

2. Work done by the gas

We know that, $\delta W = p\, dv$

On integrating from state 1 to state 2,

$$\int_1^2 \delta W = \int_1^2 p\, dv = p \int_1^2 dv$$

Or $W_{1-2} = p\,(v_2 - v_1) = mR\,(T_2 - T_1)$

3. Change in internal energy

We know that change in internal energy, $dU = m\, c_v\, dT$

On integrating from state 1 to state 2,

$$\int_1^2 dU = m\, c_v \int_1^2 dT$$

Or $$U_2 - U_1 = m\, c_v\,(T_2 - T_1)$$

4. Heat supplied or heat transfer

We know that, $\delta Q = dU + \delta W$

On integrating from state 1 to state 2,

$$\int_1^2 \delta Q = \int_1^2 dU + \int_1^2 \delta W$$

Or $$\begin{aligned} Q_{1-2} &= (U_2 - U_1) + W_{1-2} \\ &= m\, c_v\,(T_2 - T_1) + m\, R\,(T_2 - T_1) \\ &= m\,(T_2 - T_1)\,(c_v + R) \\ &= m\, c_p (T_2 - T_1) \qquad \ldots\ldots\ldots (\because c_p - c_v = R) \end{aligned}$$

5. Change in enthalpy

We know that change in enthalpy, $dH = dU + d\,(pv)$

On integrating from state 1 to state 2,

$$\int_1^2 dH = \int_1^2 dU + \int_1^2 d\,(pv)$$

Or $$\begin{aligned} H_2 - H_1 &= (U_2 - U_1) + (p_2 v_2 - p_1 v_1) \\ &= m\, c_v\,(T_2 - T_1) + m\, R\,(T_2 - T_1) \\ &= m\,(T_2 - T_1)\,(c_v + R) \\ &= m\, c_p (T_2 - T_1) \qquad \ldots\ldots\ldots (\because c_p - c_v = R) \end{aligned}$$

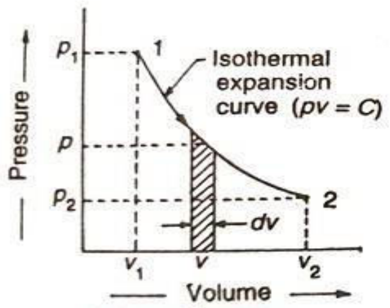
We see that change in enthalpy is equal to the heat supplied or heat transferred.

Isothermal Process (Constant Temperature Process): This process is governed by Boyle's Law.

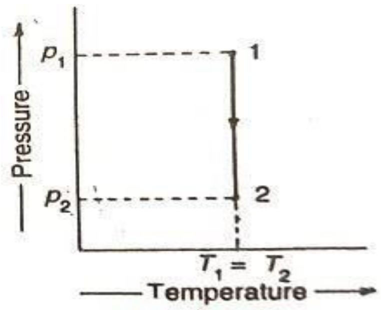
- A process in which the temperature of the working substance remains constant during its expansion or compression is called constant temperature process or isothermal process.
- It is thus obvious that in a isothermal process:
  a) There is no change in temperature.
  b) There is no change in internal energy.
  c) There is no change in enthalpy.

Now consider m kg of a certain gas, being heated at constant temperature from an initial state 1 to a final state 2.

Let p_1, v_1 and T_1 = Pressure, volume and temperature of the gas at the initial state 1.
p_2, v_2 and T_2 = Pressure, volume and temperature of the gas at the final state 2.



(a) p-v diagram.



(b) p-T diagram.

Now let us derive the following relations for the reversible isothermal process.

1. Pressure-volume-temperature (p-v-T) relationship

We know that the general gas equation is

$$\frac{p_1 v_1}{T_1} = \frac{p_2 v_2}{T_2}$$

Since the gas is heated at constant temperature, therefore $T_1 = T_2$

$$p_1 v_1 = p_2 v_2 \quad or \quad pv = Constant$$

Thus, the isothermal process is governed by Boyle's Law.

2. Work done by the gas

We know that, $\delta W = p\ dv$

On integrating from state 1 to state 2,

$$\int_1^2 \delta W = \int_1^2 p\ dv \quad or \quad W_{1-2} = \int_1^2 p\ dv \qquad \ldots\ldots\ldots (i)$$

Since the expansion of the gas is isothermal *i.e.* $p\,v = C$, therefore

$$pv = p_1 v_1 \quad or \quad p = \frac{p_1 v_1}{v}$$

Substituting this value of p in equation (i), we have

$$W_{1-2} = \int_{v_1}^{v_2} \frac{p_1 v_1}{v} dv = p_1 v_1 \int_{v_1}^{v2} dv$$

$$= p_1 v_1 [\log_e v]_{v_1}^{v2} = p_1 v_1 \log_e \left(\frac{v_2}{v_1}\right)$$

$$= 2.3\ p_1 v_1 \log\left(\frac{v_2}{v_1}\right) = 2.3\ p_1 v_1\ \log(r) \qquad \ldots\ldots\ldots\ldots (ii)$$

Where, $\boldsymbol{r = \frac{v_2}{v_1}}$, ***and is known as expansion ratio***.

We know that $p_1 v_1 = p_2 v_2 = mRT$

So the equation (ii) may also be written as follows:

$$W_{1-2} = 2.3\ mRT \log\left(\frac{v_2}{v_1}\right) = 2.3\ mRT\ \log(r)$$

Since $p_1 v_1 = p_2 v_2$, $therefore\ \frac{v_2}{v_1} = \frac{p_1}{p_2}$

$$\therefore W_{1-2} = 2.3\ mRT \log\left(\frac{p_1}{p_2}\right)$$

3. Change in internal energy

We know that change in internal energy, $dU = U_2 - U_1 = m\, c_v\, (T_2 - T_1)$

Since it is a isothermal process, *i.e.* $T_1 = T_2$, therefore

$$dU = U_2 - U_1 = 0 \quad or \quad U_1 = U_2$$

4. Heat supplied or heat transfer

We know that, $Q_{1-2} = dU + W_{1-2} = W_{1-2}$ …… ($\because$ dU = 0)

This shows that total heat supplied to the gas is equal to the work done by the gas.

5. Change in enthalpy

We know that change in enthalpy, $dH = H_2 - H_1 = m\, c_p (T_2 - T_1)$

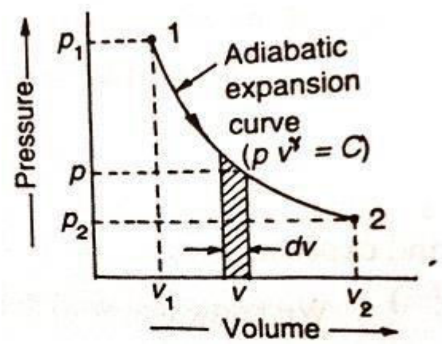
Since it is a isothermal process, *i.e.* $T_1 = T_2$, therefore

$$dH = H_2 - H_1 = 0 \quad or \quad H_1 = H_2$$

Adiabatic Process (Isentropic Process): A process, in which the working substance neither receives nor gives out heat to its surroundings, during its expansion or compression, is called an adiabatic process. This will happen when the working substance remains thermally insulated, so that no heat enters or leaves it during the process.

- It is thus obvious that, in an adiabatic or isentropic process:
  a) No heat leaves or enters the gas.
  b) The temperature of the gas changes, as the work is done at the cost of internal energy.
  c) The change in internal energy is equal to the work done.

Now consider m kg of a certain gas, being heated adiabatically from an initial state 1 to a final state 2.
Let p_1, v_1 and T_1 = Pressure, volume and temperature of the gas at the initial state 1.
p_2, v_2 and T_2 = Pressure, volume and temperature of the gas at the final state 2.



Now let us derive the following relations for a reversible adiabatic process.

1. Pressure-volume-temperature (p-v-T) relationship

Adiabatic expansion of the gas follows the law, $\boldsymbol{pv^\gamma = C \quad or \quad p_1 v_1^\gamma = p_2 v_2^\gamma = \cdots = C}$

- $\frac{p_1}{p_2} = \left(\frac{v_2}{v_1}\right)^{\gamma}$
- $\frac{T_1}{T_2} = \left(\frac{v_2}{v_1}\right)^{\gamma-1}$
- $\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}$

2. Work done during adiabatic expansion

We know that, $\delta W = p\,dv$

On integrating from state 1 to state 2,

$$\int_1^2 \delta W = \int_1^2 p\,dv \quad or \quad W_{1-2} = \int_1^2 p\,dv \qquad \dots\dots\dots(i)$$

Since adiabatic expansion of the gas follows the law,

$$pv^\gamma = p_1 v_1^\gamma \quad or \quad p = \frac{p_1 v_1^\gamma}{v^\gamma}$$

Substituting this value of *p* in equation (*i*), we have

$$W_{1-2} = \int_1^2 \frac{p_1 v_1^\gamma}{v^\gamma}\,dv = p_1 v_1^\gamma \int_1^2 v^{-\gamma}\,dv$$

$$= p_1 v_1^\gamma \left(\frac{v^{-\gamma+1}}{-\gamma+1}\right)_1^2 = \frac{p_1 v_1^\gamma}{1-\gamma}\left[v_2^{1-\gamma} - v_1^{1-\gamma}\right]$$

$$= \frac{p_1 v_1^\gamma v_2^{1-\gamma} - p_1 v_1^\gamma v_1^{1-\gamma}}{1-\gamma}$$

$$= \frac{p_2 (v_2^\gamma v_2^{1-\gamma}) - p_1 (v_1^\gamma v_1^{1-\gamma})}{1-\gamma} \qquad \dots\dots(p_1 v_1^\gamma = p_2 v_2^\gamma)$$

$$= \frac{p_2 v_2 - p_1 v_1}{1-\gamma}$$

$$= \frac{p_1 v_1 - p_2 v_2}{\gamma - 1} \qquad \dots\dots\ For\ expansion$$

$$= \frac{p_2 v_2 - p_1 v_1}{\gamma - 1} \qquad \dots\dots\ For\ compression$$

The above equation for work done may also be expressed as:

$$W_{1-2} = \frac{mR(T_1 - T_2)}{\gamma - 1} \qquad \dots\dots For\ expansion$$

$$= \frac{mR(T_2 - T_1)}{\gamma - 1} \qquad \dots\dots For\ compression$$

3. Change in internal energy

We know that change in internal energy, $dU = U_2 - U_1 = m\,c_v\,(T_2 - T_1)$

4. Heat supplied or heat transfer

We know that heat supplied or heat transferred in case of adiabatic process is zero, therefore

$$Q_{1-2} = 0$$

5. Change in enthalpy

We know that change in enthalpy, $dH = H_2 - H_1 = m\,c_p(T_2 - T_1)$

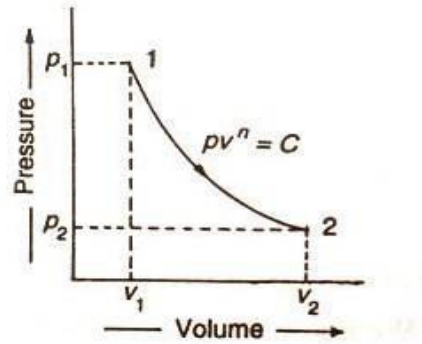
<u>Polytropic Process</u>: The polytropic process follows the general law for the expansion and compression of gases, and is given by the relation:

$$pv^n = C \quad or \quad p_1v_1^n = p_2v_2^n = \cdots = C$$

Where n is a polytropic index, which may have any value from zero to infinity.

Now consider m kg of a certain gas, being heated polytropically from an initial state 1 to a final state 2.

Let p_1, v_1 and T_1 = Pressure, volume and temperature of the gas at the initial state 1.
p_2, v_2 and T_2 = Pressure, volume and temperature of the gas at the final state 2.



Now let us derive the following relations for the polytropic process.

1. Pressure-volume-temperature (p-v-T) relationship

- $\frac{p_1}{p_2} = \left(\frac{v_2}{v_1}\right)^n$
- $\frac{T_1}{T_2} = \left(\frac{v_2}{v_1}\right)^{n-1}$
- $\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{\frac{n-1}{n}}$

2. Work done during polytropic expansion

The equation for the work done during a polytropic process may also be expressed by changing the index n for γ in the adiabatic process.

∴ Work done during a polytropic process from state 1 to state 2,

$$W_{1-2} = \frac{p_1v_1 - p_2v_2}{n-1} = \frac{mR(T_1 - T_2)}{n-1} \quad \ldots\ldots For\ expansion$$

$$= \frac{p_2v_2 - p_1v_1}{n-1} = \frac{mR(T_2 - T_1)}{n-1} \quad \ldots\ldots For\ compression$$

3. Change in internal energy

We know that change in internal energy, $dU = U_2 - U_1 = m\,c_v\,(T_2 - T_1)$

4. Heat supplied or heat transfer

We know that the heat supplied or heat transferred,

$$Q_{1-2} = W_{1-2} + dU$$

$$= \frac{p_1v_1 - p_2v_2}{n-1} + m\,c_v\,(T_2 - T_1)$$

$$= \frac{mR(T_1 - T_2)}{n-1} + m \times \frac{R}{\gamma - 1}(T_2 - T_1)$$

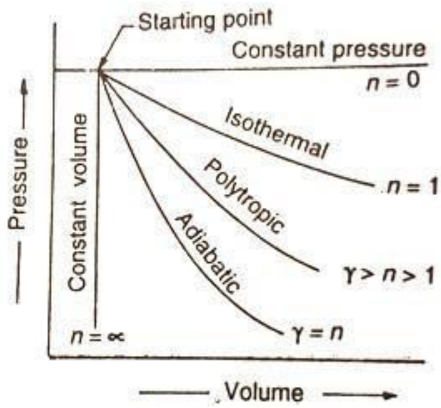
$$= mR(T_1 - T_2)\left[\frac{1}{n-1} - \frac{1}{\gamma-1}\right]$$
$$= mR(T_1 - T_2)\left[\frac{(\gamma-1)-(n-1)}{(n-1)(\gamma-1)}\right]$$
$$= mR(T_1 - T_2)\left[\frac{\gamma-n}{(n-1)(\gamma-1)}\right]$$
$$= \frac{\gamma-n}{\gamma-1} \times \frac{mR(T_1-T_2)}{n-1}$$

5. Change in enthalpy

We know that change in enthalpy, $dH = H_2 - H_1 = m\, c_p(T_2 - T_1)$

<u>General Laws for Expansion and Compression</u>:

The general law of expansion or compression of a perfect gas is pv^n = Constant. It gives the relationship between pressure and volume of a given quantity of gas. The value of ***'n'*** may be varies between zero and infinity.

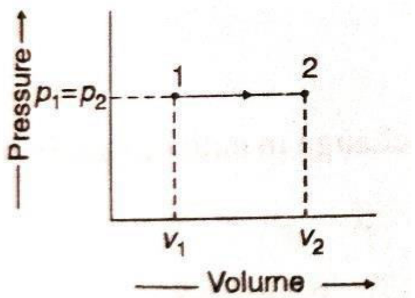


<u>SOLVE SIMPLE PROBLEMS ON ABOVE</u>:

<u>Example 3.6</u>: The values of specific heats at constant pressure and at constant volume for an ideal gas are 0.984 kj/kg K and 0.728 kj/kg K. Find the values of characteristic gas constant (R) and ratio of specific heats (γ) for the gas. If one kg of this gas is heated at constant pressure from 25^0 C to 200^0 C, estimate the heat added, ideal work done and change in internal energy. Also calculate the pressure and final volume, if the initial volume was 2 m^3.

<u>Solution</u>: Data given: $c_p = 0.984$ kj/kg K, $c_v = 0.728$ kj/kg K, $m = 1$ kg, $T_1 = 25^0$ C $= 25 + 273 = 298$ K, $T_2 = 200^0$ C $= 200 + 273 = 473$ K, $v_1 = 2\ m^3$

The heating of gas at constant pressure is shown in figure below



(a) p-v diagram.

Characteristic gas constant

We know that characteristic gas constant,

$$\begin{aligned} R &= c_p - c_v \\ &= 0.984 - 0.728 \\ &= 0.256\ kj/kg\ K \end{aligned}$$

Ratio of specific heats

We know that ratio of specific heats,

$$\begin{aligned} \gamma &= \frac{c_p}{c_v} \\ &= \frac{0.984}{0.728} = 1.35 \end{aligned}$$

Heat added

We know that heat added during constant pressure operation,

$$\begin{aligned} Q_{1-2} &= mc_p(T_2 - T_1) \\ &= 1 \times 0.984(473 - 298) \\ &= 172.2\ kj \end{aligned}$$

Work done

We know that work done during constant pressure operation,

$$\begin{aligned} W_{1-2} &= p(v_2 - v_1) \\ &= m\,R(T_2 - T_1) \\ &= 1 \times 0.256(473 - 298) = 44.8\ kj \end{aligned}$$

Change in internal energy

We know that change in internal energy,

$$\begin{aligned} dU &= mc_v(T_2 - T_1) \\ &= 1 \times 0.728(473 - 298) = 127.4\ kj \end{aligned}$$

Pressure and final volume of the gas if the initial volume, v_1 = 2 m^3

Let $p_1 = p_2$ = Pressure of the gas, and
v_2 = Final volume of the gas.

We know that,

$$\frac{p_1 v_1}{T_1} = \frac{p_2 v_2}{T_2} \quad or \quad \frac{v_1}{T_1} = \frac{v_2}{T_2}$$

$$\therefore\ v_2 = \frac{v_1 T_2}{T_1} = \frac{2 \times 473}{298} = 3.17\ m^3$$

We also know that,

$$p_1 v_1 = m\,R\,T_1$$

$$\begin{aligned} \therefore p_1 &= \frac{m\,R\,T_1}{v_1} \\ &= \frac{1 \times 256 \times 298}{2} = 38140\ N/m^2 \\ &= 0.3814\ bar \end{aligned}$$

Example 3.6. *A quantity of air has a volume of 0.4 m^3 at a pressure of 5 bar and a temperature of 80° C. It is expanded in a cylinder at a constant temperature to a pressure of 1 bar. Determine the amount of work done by the air during expansion.*

Solution. Given : $v_1 = 0.4\ m^3$; $p_1 = 5$ bar $= 0.5 \times 10^6$ N/m^2 ; $*T = 80°$ C ; $p_2 = 1$ bar $= 0.1 \times 10^6$ N/m^2

First of all, let us find the volume of air at the end of expansion (*i.e.* v_2). We know that

$$p_1 v_1 = p_2 v_2 \quad \text{or} \quad v_2 = \frac{p_1 v_1}{p_2} = \frac{0.5 \times 10^6 \times 0.4}{0.1 \times 10^6} = 2\ m^3$$

and expansion ratio, $r = v_2 / v_1 = 2 / 0.4 = 5$

We know that workdone by the air during expansion,

$$W_{1-2} = 2.3\, p_1 v_1 \log r = 2.3 \times 0.5 \times 10^6 \times 0.4 \log 5\ J$$

$$= 0.46 \times 10^6 \times 0.699 = 321\ 540\ J = 321.54\ kJ \ \textbf{Ans.}$$

Example 3.7. *0.1 m^3 of air at a pressure of 1.5 bar is expanded isothermally to 0.5 m^3. Calculate the final pressure of the gas and heat supplied during the process.*

Solution. Given : $v_1 = 0.1\ m^3$; $p_1 = 1.5$ bar $= 0.15 \times 10^6$ N/m^2 ; $v_2 = 0.5\ m^3$

Final pressure of the gas

Let p_2 = Final pressure of the gas.

We know that $p_1 v_1 = p_2 v_2$

$$\therefore \quad p_2 = \frac{p_1 v_1}{v_2} = \frac{0.15 \times 10^6 \times 0.1}{0.5} = 0.03 \times 10^6\ \text{N/m}^2 = 0.3\ \text{bar} \ \textbf{Ans.}$$

Heat supplied during the process

We know that expansion ratio,

$$r = v_2 / v_1 = 0.5 / 0.1 = 5$$

$\therefore$ Workdone during the process,

$$W_{1-2} = 2.3\, p_1 v_1 \log r = 2.3 \times 0.15 \times 10^6 \times 0.1 \log 5\ J$$

$$= 0.0345 \times 10^6 \times 0.699 = 24\ 115\ J = 24.115\ kJ$$

We know that in an isothermal process, heat supplied (Q_{1-2})is equal to the workdone during the process.

$$\therefore \quad Q_{1-2} = W_{1-2} = 24.115\ kJ \ \textbf{Ans.}$$

Example 3.8. *One litre of hydrogen at 0° C is suddenly compressed to one-half of its volume. Find the change in temperature of the gas, if the ratio of two specific heats for hydrogen is 1.4.*

Solution. Given : v_1 = 1 litre ; $T_1 = 0^\circ \text{ C} = 0 + 273 = 273$ K ; $v_2 = v_1/2 = 1/2 = 0.5$ litre ; $\gamma = 1.4$

Let $\quad T_2$ = Final temperature of the gas.

We know that
$$\frac{T_1}{T_2} = \left(\frac{v_2}{v_1}\right)^{\gamma - 1} = \left(\frac{0.5}{1}\right)^{1.4-1} = (0.5)^{0.4} = 0.758$$

$$\therefore \quad T_2 = T_1 / 0.758 = 273 / 0.758 = 360.16 \text{ K}$$

$$= 360.16 - 273 = 87.16^\circ \text{ C} \quad \textbf{Ans.}$$

Example 3.9. *The initial volume of 0.18 kg of a certain gas was 0.15 m^3 at a temperature of 15° C and a pressure of 1 bar. After adiabatic compression to 0.056 m^3, the pressure was found to be 4 bar. Find ;*

1. Gas constant ; 2. Molecular mass of the gas ; 3. Ratio of specific heats ; 4. Two specific heats, one at a constant pressure and the other at a constant volume ; and 5. Change of internal energy.

Solution. Given : $m = 0.18$ kg ; $v_1 = 0.15 \text{ m}^3$; $T_1 = 15^\circ \text{ C} = 15 + 273 = 288$ K ; p_1 = 1 bar $= 0.1 \times 10^6 \text{ N/m}^2$; $v_2 = 0.056 \text{ m}^3$; $p_2 = 4 \text{ bar} = 0.4 \times 10^6 \text{ N/m}^2$

The p-v diagram is shown in Fig. 3.11.

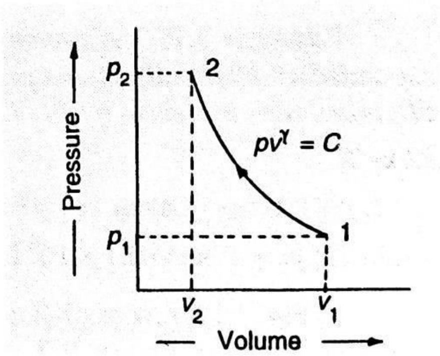


Fig. 3.11

1. *Gas constant*

Let $\quad R$ = Gas constant.

We know that
$$p_1 v_1 = m R T_1$$

$$\therefore \quad R = \frac{p_1 v_1}{m T_1} = \frac{0.1 \times 10^6 \times 0.15}{0.18 \times 288}$$

$$= 289.4 \text{ J/kg K} = 0.2894 \text{ kJ/kg K} \quad \textbf{Ans.}$$

2. *Molecular mass of the gas*

We know that molecular mass of the gas,
$$M = \frac{\text{Universal gas constant } (R_u)}{\text{Characteristic gas constant } (R)} = \frac{8314}{289.4} = 28.73 \text{ kg} \quad \textbf{Ans.}$$

$$\ldots (\because R_u = 8314 \text{ J/kg K, for all gases})$$

3. *Ratio of specific heats*

We know that ratio of specific heats,
$$\gamma = \frac{\log\left(\dfrac{p_2}{p_1}\right)}{\log\left(\dfrac{v_1}{v_2}\right)} = \frac{\log\left(\dfrac{0.4 \times 10^6}{0.1 \times 10^6}\right)}{\log\left(\dfrac{0.15}{0.056}\right)} = \frac{\log 4}{\log 2.678} = \frac{0.6020}{0.4278}$$

$$= 1.407 \quad \textbf{Ans.}$$

4. *Specific heat at a constant volume and constant pressure*

Let c_v = Specific heat at a constant volume, and

c_p = Specific heat at a constant pressure.

We know that $c_p - c_v = R$ or $1.407\, c_v - c_v = 0.2894$ $\quad \ldots (\because \gamma = c_p / c_v = 1.407)$

$\therefore \quad c_v = 0.2894 / 0.407 = 0.711$ kJ/kg K **Ans.**

and $\quad c_p = 1.407\, c_v = 1.407 \times 0.711 = 1$ kJ/kg K **Ans.**

5. *Change in internal energy*

First of all, let us find the final temperature (T_2). We know that

$$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{1}{4}\right)^{\frac{1.407-1}{1.407}} = (0.25)^{0.289} = 0.67$$

$\therefore \quad T_2 = T_1 / 0.67 = 288 / 0.67 = 430$ K

We know that change in internal energy,

$$dU = U_2 - U_1 = m\, c_v (T_2 - T_1) = 0.18 \times 0.711\, (430 - 288) \text{ kJ}$$

$$= 18.17 \text{ kJ } \textbf{Ans.}$$

Example 3.10. *A system contains 0.15 m³ of a gas at a pressure of 3.8 bar and 150° C. It is expanded adiabatically till the pressure falls to 1 bar. The gas is then heated at a constant pressure till its enthalpy increases by 70 kJ. Determine the total work done. Take c_p = 1 kJ/kg K and c_v = 0.714 kJ/kg K.*

Solution. Given : $v_1 = 0.15$ m³ ; $p_1 = 3.8$ bar $= 0.38 \times 10^6$ N/m² ; $T_1 = 150°$ C $= 150 + 273 = 423$ K ; $p_2 = 1$ bar $= 0.1 \times 10^6$ N/m² ; $dH = 70$ kJ ; $c_p = 1$ kJ/kg K ; $c_v = 0.714$ kJ/kg K

In Fig. 3.12, process 1-2 represents adiabatic expansion of the gas and the process 2-3 represents heating at constant pressure.

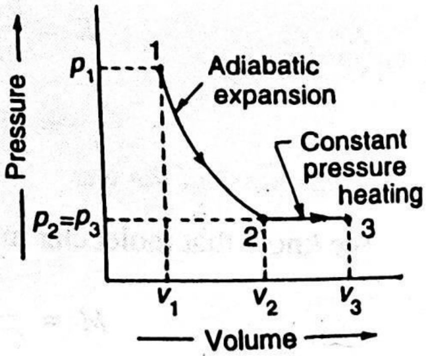


Fig. 3.12

First of all, let us find the temperature (T_2) and volume (v_2) after the adiabatic expansion.

We know that adiabatic index,

$$\gamma = c_p / c_v = 1 / 0.714 = 1.4$$

$$\therefore \quad \frac{T_1}{T_2} = \left(\frac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{3.8}{1}\right)^{\frac{1.4-1}{1.4}} = (3.8)^{0.286} = 1.465$$

or $\quad T_2 = T_1 / 1.465 = 423 / 1.465 = 288.7$ K

and

$$\frac{v_1}{v_2} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}} = \left(\frac{1}{3.8}\right)^{\frac{1}{1.4}} = (0.263)^{0.714} = 0.385$$

$\therefore \quad v_2 = v_1 / 0.385 = 0.15 / 0.385 = 0.39$ m³

Now let us find the temperature (T_3) and volume (v_3) after constant pressure heating.

Let m = Mass of gas contained in the system.

We know that gas constant,

$$R = c_p - c_v = 1 - 0.714 = 0.286 \text{ kJ/kg K} = 286 \text{ J/kg K}$$

and $$p_1 v_1 = m R T_1$$

$$\therefore \quad m = \frac{p_1 v_1}{R T_1} = \frac{0.38 \times 10^6 \times 0.15}{286 \times 423} = 0.47 \text{ kg}$$

We also know that increase in enthalpy (dH),

$$70 = m c_p (T_3 - T_2) = 0.47 \times 1 (T_3 - 288.7) \text{ kJ}$$

$$\therefore \quad T_3 = \frac{70}{0.47} + 288.7 = 437.6 \text{ K}$$

Since the heating is at constant pressure, therefore

$$\frac{v_2}{T_2} = \frac{v_3}{T_3} \quad \text{or} \quad v_3 = \frac{v_2 T_3}{T_2} = \frac{0.39 \times 437.6}{288.7} = 0.59 \text{ m}^3$$

We know that work done during adiabatic expansion,

$$W_{1-2} = \frac{p_1 v_1 - p_2 v_2}{\gamma - 1} = \frac{0.38 \times 10^6 \times 0.15 - 0.1 \times 10^6 \times 0.39}{1.4 - 1} \text{ J}$$

$$= \frac{57 \times 10^3 - 39 \times 10^3}{0.4} = 45\,000 \text{ J} = 45 \text{ kJ}$$

and workdone during constant pressure heating,

$$W_{2-3} = p_2 (v_3 - v_2) = 0.1 \times 10^6 (0.59 - 0.39) = 20\,000 \text{ J} = 20 \text{ kJ}$$

$\therefore$ Total work done, $W = W_{1-2} + W_{2-3} = 45 + 20 = 65 \text{ kJ}$ **Ans.**

Example 3.13. *A certain quantity of air has a volume of 0.028 m^3 at a pressure of 1.25 bar and 25° C. It is compressed to a volume of 0.0042 m^3 according to the law $pv^{1.3}$ = Constant. Find the final temperature and work done during compression. Also determine the reduction in pressure at a constant volume required to bring the air back to its original temperature.*

Solution. Given : $v_1 = 0.028 \text{ m}^3$; $p_1 = 1.25 \text{ bar} = 0.125 \times 10^6 \text{ N/m}^2$; $T_1 = 25° \text{ C} = 25 + 273 = 298 \text{ K}$; $v_2 = 0.0042 \text{ m}^3$; $n = 1.3$

The p-v diagram is shown in Fig. 3.16.

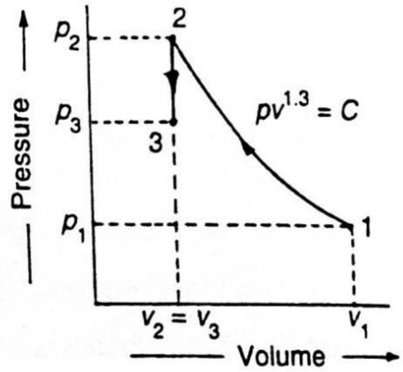


Fig. 3.16

Final temperature

Let T_2 = Final temperature.

We know that

$$\frac{T_1}{T_2} = \left(\frac{v_2}{v_1}\right)^{n-1} = \left(\frac{0.0042}{0.028}\right)^{1.3-1}$$

$$= (0.15)^{0.3} = 0.566$$

$$\therefore \quad T_2 = T_1 / 0.566 = 298 / 0.566 = 526.5 \text{ K} = 526.5 - 273 = 253.5° \text{ C} \quad \textbf{Ans.}$$

Workdone during compression

First of all, let us find the final pressure (p_2) at the end of compression. We know that

$$p_1 v_1^n = p_2 v_2^n \quad \text{or} \quad \frac{p_1}{p_2} = \left(\frac{v_2}{v_1}\right)^n = \left(\frac{0.0042}{0.028}\right)^{1.3} = 0.085$$

$$\therefore \quad p_2 = p_1 / 0.085 = 1.25 / 0.085 = 14.7 \text{ bar} = 1.47 \times 10^6 \text{ N/m}^2$$

We know that workdone during compression,

$$W_{1-2} = \frac{p_2 v_2 - p_1 v_1}{n-1} = \frac{1.47 \times 10^6 \times 0.0042 - 0.125 \times 10^6 \times 0.028}{1.3 - 1}$$

$$= \frac{6174 - 3500}{0.3} = 8913 \text{ J} = 8.913 \text{ kJ} \quad \textbf{Ans.}$$

Pressure at a constant volume

Let p_3 = Pressure at a constant volume required to bring the air back to its initial temperature, $T_1 = 298$ K.

We know that for a constant volume process 2-3,

$$\frac{p_2}{T_2} = \frac{p_3}{T_3} \quad \text{or} \quad p_3 = \frac{p_2 T_3}{T_2} = \frac{14.7 \times 298}{526.5} = 8.32 \text{ bar} \quad \textbf{Ans.}$$

$$\ldots (\because v_2 = v_3 \text{ and } T_3 = T_1)$$

Example 3.14. *A gas mixture obeying perfect gas law has a molecular mass of 26.7. The gas mixture is compressed through a compression ratio of 12 according to the law $pv^{1.25}$ = Constant, from initial conditions of 0.9 bar and 333 K. Assuming a mean molar specific heat at constant volume of 21.1 kJ/kg K, find, per kg of mass, the workdone and heat flow across the cylinder walls.*

For the above gas, determine the value of characteristic gas constant, molar specific heat at a constant pressure and ratio of specific heats.

Solution. Given : $M = 26.7$; $r = v_1/v_2 = 12$; $n = 1.25$; $p_1 = 0.9$ bar $= 0.09 \times 10^6$ N/m^2 ; $T_1 = 333$ K ; $c_{vm} = 21.1$ kJ/kg K ; $m = 1$ kg

The p-v diagram is shown in Fig. 3.17.

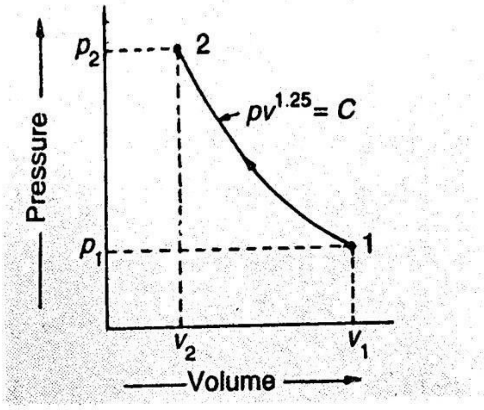


Fig. 3.17

Workdone per kg of gas

First of all, let us find the initial volume (v_1), final volume (v_2) and final pressure (p_2).

We know that $p_1 v_1^n = p_2 v_2^n$

$$\therefore \quad p_2 = p_1 \left(\frac{v_1}{v_2}\right)^n = 0.9\,(12)^{1.25}$$

$$= 20.1 \text{ bar} = 2.01 \times 10^6 \text{ N/m}^2$$

We also know that gas constant,

$$R = \frac{R_u}{M} = \frac{8314}{26.7} = 311.4 \text{ J/kg K}$$

and

$$p_1 v_1 = m\,R\,T_1 \quad \text{or} \quad v_1 = \frac{m\,R\,T_1}{p_1} = \frac{1 \times 311.4 \times 333}{0.09 \times 10^6} = 1.15 \text{ m}^3$$

$$\therefore \quad v_2 = v_1/12 = 1.15/12 = 0.096 \text{ m}^3 \qquad \ldots (\because v_1/v_2 = 12)$$

We know that workdone

$$W_{1-2} = \frac{p_1 v_1 - p_2 v_2}{n-1} = \frac{0.09 \times 10^6 \times 1.15 - 2.01 \times 10^6 \times 0.096}{1.25 - 1}$$

$$= \frac{103\,500 - 192\,960}{0.25} = -357\,840 \text{ J} = -357.84 \text{ kJ} \quad \textbf{Ans.}$$

The *negative* sign indicates that the work is done on the gas.

Heat flow across the cylinder walls

Let T_2 = Final temperature.

We know that $$\frac{T_1}{T_2} = \left(\frac{v_2}{v_1}\right)^{n-1} = \left(\frac{1}{12}\right)^{1.25-1} = 0.537$$

$$\therefore \quad T_2 = T_1 / 0.537 = 333 / 0.537 = 620 \text{ K}$$

and specific heat at constant volume,

$$c_v = c_{vm} / M = 21.1 / 26.7 = 0.79 \text{ kJ/kg K}$$

$\therefore$ Change in internal energy,

$$dU = U_2 - U_1 = m\, c_v (T_2 - T_1) = 1 \times 0.79\,(620 - 333) = 226.7 \text{ kJ}$$

We know that heat flow across the cylinder walls,

$$Q_{1-2} = W_{1-2} + dU = -357.84 + 226.7 = -131.1 \text{ kJ} \quad \textbf{Ans.}$$

The *negative* sign indicates that the heat is rejected through the cylinder walls.

Characteristic gas constant

We know that characteristic gas constant,

$$R = \frac{\text{Universal gas constant}}{\text{Molecular mass}} = \frac{R_u}{M} = \frac{8314}{26.7}$$

$$\ldots (\because R_u = 8314 \text{ J/kg K, for all gases})$$

$$= 311.4 \text{ J/kg K} = 0.3114 \text{ kJ/kg K} \quad \textbf{Ans.}$$

Molar specific heat at a constant pressure

Let c_{pm} = Molar specific heat at a constant pressure.

We know that $c_p - c_v = R$ or $c_p - 0.79 = 0.3114$

$$\therefore \quad c_p = 0.3114 + 0.79 = 1.1014 \text{ kJ/kg K}$$

and $$c_{pm} = M\, c_p = 26.7 \times 1.1014 = 29.4 \text{ kJ/kg K} \quad \textbf{Ans.}$$

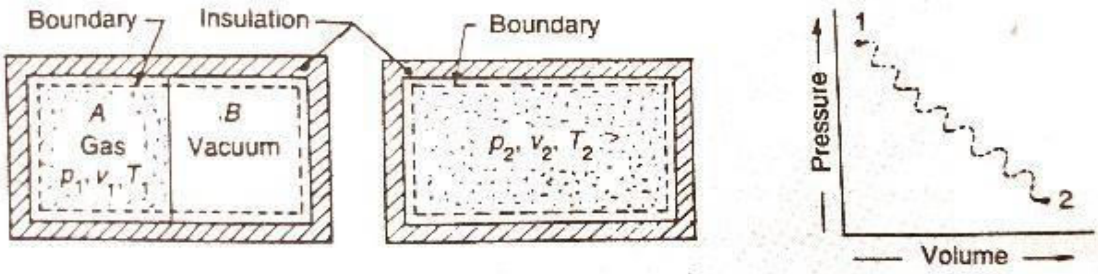
Ratio of specific heats

We know that ratio of specific heats,

$$\gamma = c_p / c_v = 1.1014 / 0.79 = 1.394 \quad \textbf{Ans.}$$

<u>FREE EXPANSION & THROTTLING PROCESS</u>:

<u>Free Expansion Process</u>: The free expansion process is an irreversible non-flow process. A free expansion occurs when a fluid is allowed to expand suddenly into a vacuum chamber through an orifice of large dimensions.



(*a*) Before expansion (*b*) After expansion. (*c*) *p-v* diagram.

Consider two chambers ***A*** and ***B*** separated by a partition as shown in the figure (*a*). Let the chamber ***A*** contains a perfect gas having volume v_1, pressure p_1, and temperature T_1, and the chamber ***B*** is completely evacuated. These chambers are perfectly insulated so that no heat transfer takes place from or to its surroundings. Now, if the partition is removed, the gas will expand freely and occupy the whole space as shown in the figure (*b*).

- Since the system is perfectly insulated, therefore no heat transfer takes place between the system and surroundings i.e. $Q_{1-2} = 0$.
- Since there is no expansion of the boundary of the system, because it is rigid, therefore no work is done i.e. $W_{1-2} = 0$.
- We know that, $Q_{1-2} = dU + W_{1-2}$

  ⇨ $dU = 0$

  ⇨ $U_2 = U_1$
- We know that, $dU = mc_v(T_2 - T_1)$

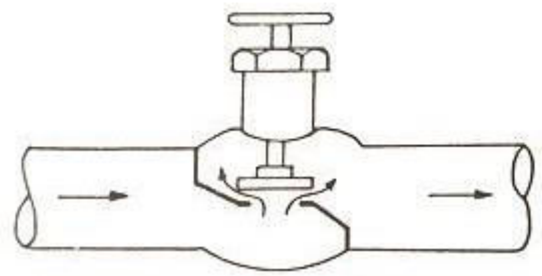
  ⇨ $0 = mc_v(T_2 - T_1)$

  ⇨ $T_2 = T_1$
- We know that, $dH = mc_p(T_2 - T_1)$

  ⇨ $dH = 0$

  ⇨ $H_2 = H_1$

Throttling Process:



The throttling process is an irreversible steady flow expansion process in which a perfect gas is expanded through a slightly opened valve as shown in the figure above. Due to the fall in pressure during expansion, the gas should come out with a large velocity, but due to high frictional resistance between the gas and the wall of the aperture, there is no considerable change in velocity. The kinetic energy of the gas is converted into heat which is utilised in warming the gas to its initial temperature. Since no heat is supplied or rejected during the throttling process, and also no work is done, therefore

$$q_{1-2} = 0 \quad and \quad w_{1-2} = 0$$

We know that steady flow energy equation for unit mass flow is

$$h_1 + \frac{V_1^2}{2} + gz_1 + q_{1-2} = h_2 + \frac{V_2^2}{2} + gz_2 + w_{1-2}$$

Since there is no considerable change in velocity and the inlet and outlet are at the same level, therefore $V_1 = V_2$ and $z_1 = z_2$. Now the steady flow energy equation becomes,

$$h_1 = h_2$$

IMPORTANT QUESTIONS:

Short type questions:

1. State Boyle's Law.

Ans: This law was formulated by Robert Boyle in 1662. It states, ***"The absolute pressure of a given mass of a perfect gas varies inversely as its volume, when the temperature remains constant".***

Mathematically, $p \propto \frac{1}{v} \quad or \quad pv = constant$

$$or\ p_1v_1 = p_2v_2 = p_3v_3 = \cdots = constant$$

2. State Charles's Law.

Ans: This law was formulated by a Frenchman Jacques A.C. Charles in about 1787. It states, ***"The volume of a given mass of a perfect varies directly as its absolute temperature, when the absolute pressure remains constant".***

Mathematically, $\boldsymbol{v \propto T \quad or \quad \frac{v}{T} = constant}$

$$or\ \frac{v_1}{T_1} = \frac{v_2}{T_2} = \frac{v_3}{T_3} = \cdots = constant$$

3. State Gay-Lussac Law.

Ans: This law states, ***"The absolute pressure of a given mass of a perfect gas varies directly as its absolute temperature when the volume remains constant".***

Mathematically, $\boldsymbol{p \propto T \quad or \quad \frac{p}{T} = constant}$

$$or\ \frac{p_1}{T_1} = \frac{p_2}{T_2} = \frac{p_3}{T_3} = \cdots = constant$$

4. State Avogadro's Law.

Ans: This law states, ***"Equal volumes of all gases, at the same temperature and pressure, contain equal no of molecules".***

Avogadro's experiments shows that the average volume for 1 Kg-mole of any perfect gas is 22.413 m^3 at standard atmospheric pressure (1.01325 bar) and 0^0C NTP condition.

5. State Dalton's Law of Partial Pressure.

Ans: When two or more gases which don't react chemically with one another are enclosed in a vessel, then the total pressure exerted by the mixture of gases will be equal to the sum of the partial pressures which each gas would exert if present alone in that space.

Mathematically,

$$\boldsymbol{p = p_a + p_b + p_c \ldots\ldots\ldots\ldots\ldots}$$

Where, p = Total pressure of the mixture of gases

p_a, p_b, p_c = Partial pressures of each gas

6. Define specific heat of the gas.

Ans: The specific heat of a substance may be broadly defined as the amount of heat required to raise the temperature of its unit mass through one degree.

Mathematically,

$$\boldsymbol{C = \frac{Q}{m \times \Delta t}}$$

Where, C = Specific heat of a substance

Q = Amount of heat transfer

m = Mass of the substance

Δt = Rise in temperature

- S.I. unit, ***J/kg K or kJ/kg K***

<u>**Long type questions**</u>:

1. Derive the relationship between C_p and C_v .

2. Derive the general gas equation from Boyle's Law and Charles's Law.

3. **The values of specific heats at constant pressure and at constant volume for an ideal gas are 0.984 kj/kg K and 0.728 kj/kg K. Find the values of characteristic gas constant (R) and ratio of specific heats (γ) for the gas. If one kg of this gas is heated at constant pressure from 25^0 C to 200^0 C, estimate the heat added, ideal work done and change in internal energy. Also calculate the pressure and final volume, if the initial volume was 2 m^3.**

4. A system contains 0.20 m^3 of a gas at a pressure of 4.8 bar and 180^0 C. it is expanded adiabatically till the pressure falls to 1 bar. The gas is then heated at a constant pressure till its enthalpy increases by *70 kj*. Determine the total work done. Take C_p = *1 kj/kg K* and C_v = *0.714 kj/kg K*.

Hints: Refer example **3.10**.

CHAPTER 04

BENDING MOMENT& SHEAR FORCE

LEARNING OBJECTIVES:

Types of beam and load

Concepts of Shear force and bending moment

Shear Force and Bending moment diagram and its salient features illustration in cantilever beam, simply supported beam and over hanging beam under point load and uniformly distributed load.

Types of beam and load

Beam:

Beam is a structural member which is acted upon by a system of external loads at right angles to the axis.

Types of beams:

There are six types of beams.

- Cantilever beam
- Simply supported beam
- Overhanging beam
- Propped cantilever
- Fixed beam
- Continuous beam

Cantilever beam:

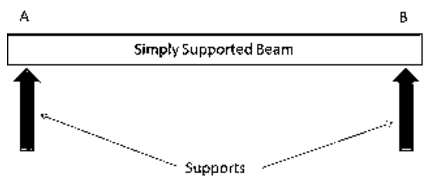
- A beam fixed at one end and free (unsupported) at the other end is called a cantilever beam or simply cantilever.

Cantilever Beam

Simply supported beam:

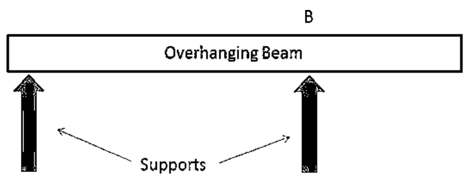
- A beam having its ends freely resting on supports is known as a simply supported beam.

Simply Supported Beam



Overhanging beam:

- A beam having its end portion extended beyond the support is known as overhanging beam.

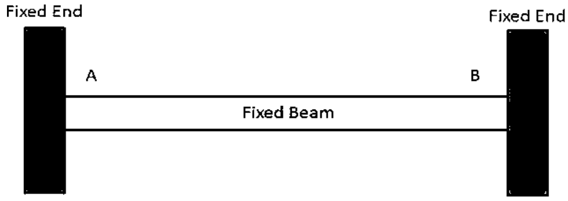


Propped cantilever:

- When a support is provided at some suitable point of a cantilever beam to resist its deflection, then it is known as propped cantilever.

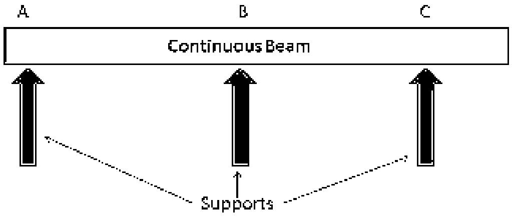
Fixed beam:

- A beam whose both ends are fixed or built-in walls is known as fixed beam.
- It is also called a built-in beam or encastred beam.



Continuous beam:

- When more than two supports are provided for a beam, it is known as continuous beam.
- The supports at the extreme left and right are called end supports and all the other supports are called intermediate supports.



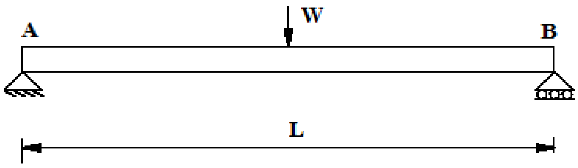
Types of loads:

A beam may be subjected to either or in combination of the following types of loads:

1. Concentrated or point load,
2. Uniformly distributed load and
3. Uniformly varying load

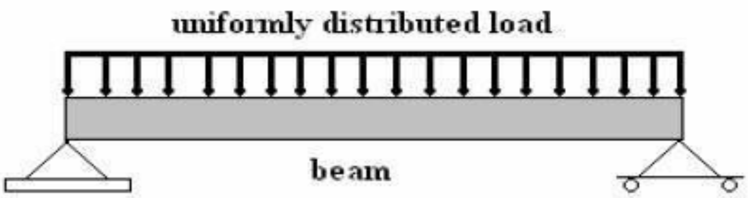
Concentrated or point load:

- It is assumed to act at a point. Practically it is applied over a small area.



Uniformly distributed load:

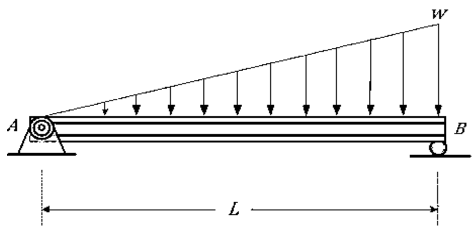
- It is distributed (or spread) uniformly over some length. The intensity of load is constant.



Uniformly varying load:

- It is distributed uniformly over some length of beam but the intensity of load varies.

- The load varies from some value at a position to some other value at another position on the beam in such a way that the change in load per unit length is same over loaded portion of the beam.



Concepts of Shear force and bending moment

Shear Force:

- The shear force (briefly written as S.F.) at the cross-section of a beam may be defined as the unbalanced vertical force to the right or left of the section.

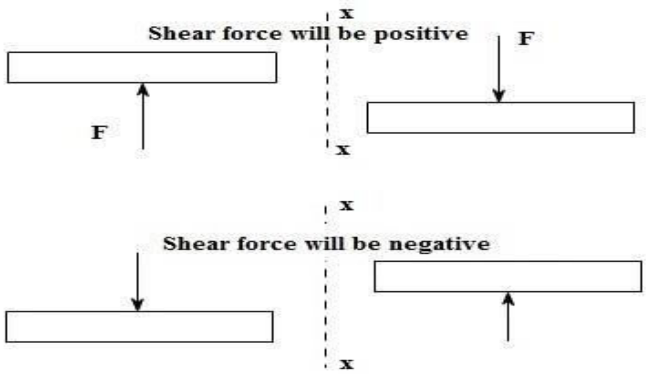
Bending Moment:

- The bending moment (briefly written as B.M.) at the cross-section of a beam may be defined as the algebraic sum of the moments of the forces, to the right or left of the section.

Sign Conventions:

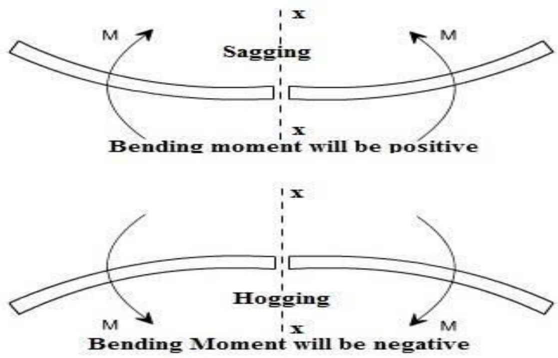
For shear force:

- All the upward forces to the left of the section cause positive shear and those acting downward cause negative shear.
- All the downward forces to the right of the section cause positive shear and those acting upward cause negative shear.



For bending moment:

- When bending moment is calculated using the loads acting to the left side of the section, clockwise moment is positive and anticlockwise moment is negative.
- When bending moment is calculated using the loads acting to the right side of the section, anticlockwise moment is positive and clockwise moment is negative
- The positive bending moment is often called sagging bending moment and negative as hogging bending moment.



Shear Force and Bending moment diagram and its salient features illustration in cantilever beam, simply supported beam and over hanging beam under point load and uniformly distributed load

S.F.D: A shear force diagram is one which shows the variation of shear force along the length of the beam.

B.M.D: A bending moment diagram is one which shows the variation of bending moment along the length of the beam.

Note: While drawing the shear force or bending moment diagrams, all the positive values are plotted above the base line and negative values below it.

Relation between Loading, Shear Force and Bending Moment:

The following relations between loading, shear force and bending moment at a point or between any two sections of a beam are important from the subject point of view:

- If there is a point load at a section on the beam, then the shear force suddenly changes (i.e., the shear force line is vertical). But the bending moment remains the same.
- If there is no load between two points, then the shear force does not change (i.e., shear force line is horizontal). But the bending moment changes linearly (i.e., bending moment line is an inclined straight line).
- If there is a uniformly distributed load between two points, then the shear force changes linearly (i.e., shear force line is an inclined straight line). But the bending moment changes according to the parabolic law. (i.e., bending moment line will be a parabola).
- If there is a uniformly varying load between two points then the shear force changes according to the parabolic law (i.e., shear force line will be a parabola). But the bending moment changes according to the cubic law.

Cantilever with a Point Load at its Free End:

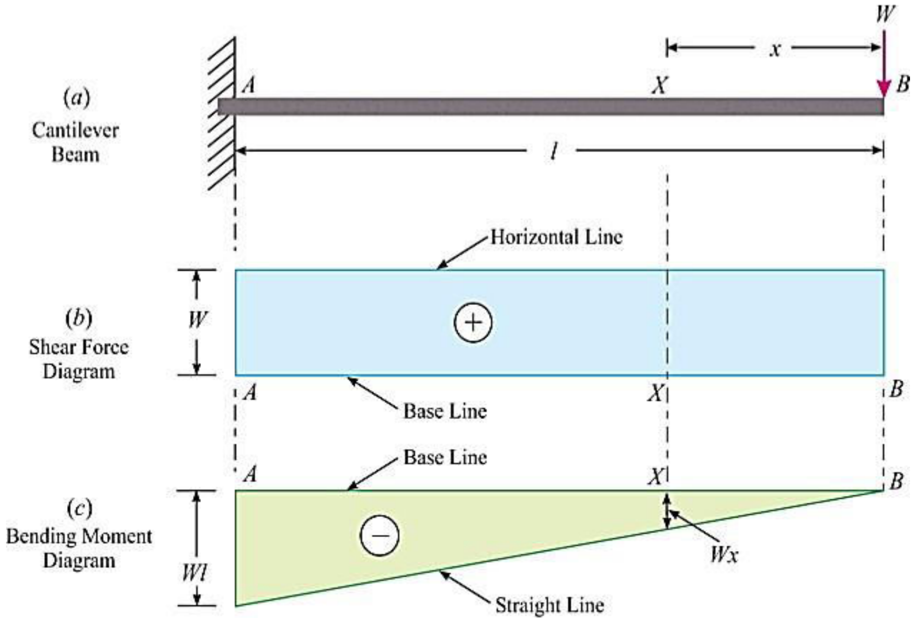
- Consider a cantilever AB of length l and carrying a point load W at its free end B as shown in Figure. We know that shear force at any section X, at a distance x from the free end, is equal to the total unbalanced vertical force. i.e.,

$$F_x = W$$

- and bending moment at this section,

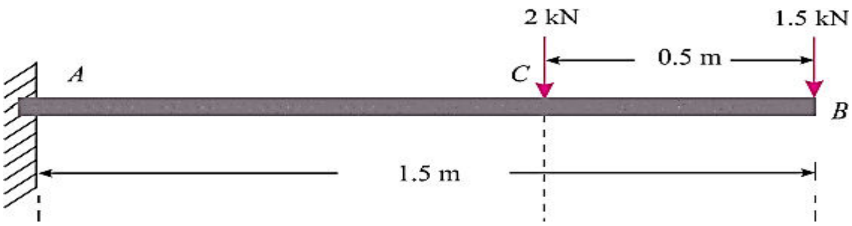
$$M_x = -W.x$$

$$\dots\dots (Minus\ sign\ due\ to\ hogging)$$



- Thus, from the equation of shear force, we see that the shear force is constant and is equal to W at all sections between B and A. And from the bending moment equation, we see that the bending moment is zero at B (where $x = 0$) and increases by a straight-line law to $-Wl$; at (where $x = l$).

Example – 1: Draw shear force and bending moment diagrams for a cantilever beam of span 1.5 m carrying point loads as shown in Figure.



Solution. Given: Span (l) = 1.5 m; Point load at B (W_1) = 1.5 kN and point load at C (W_2) = 2 kN.

Shear force diagram

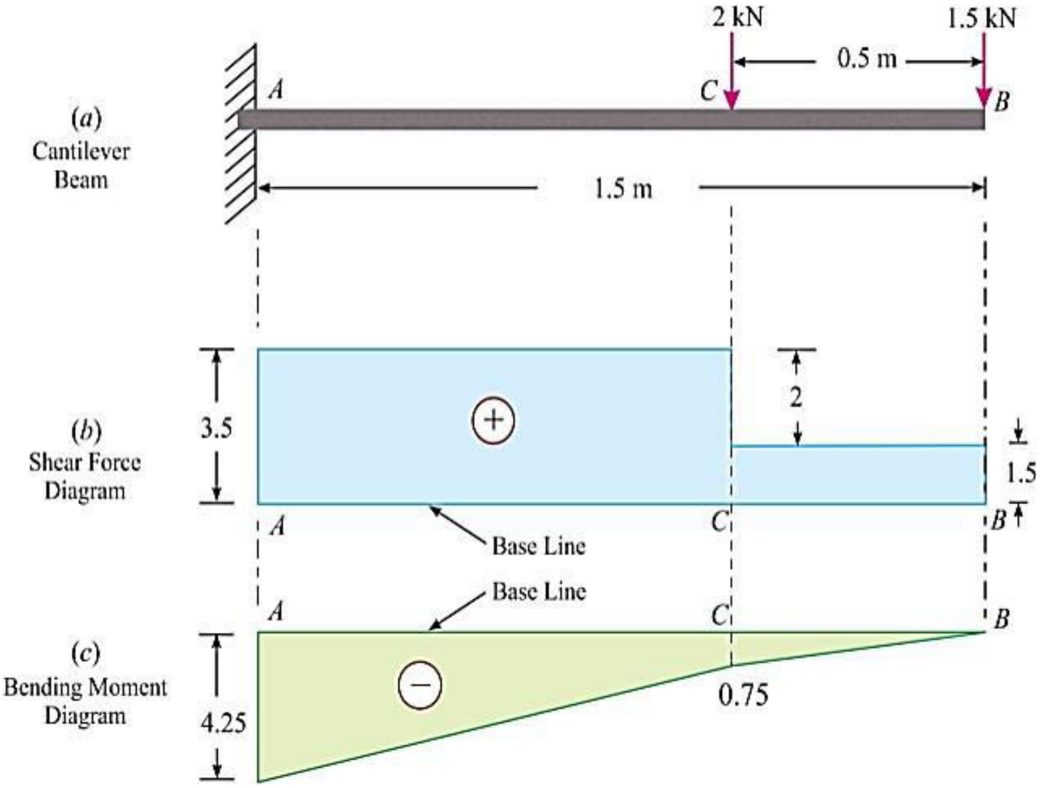
The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_B = W_1 = 1.5\ kN$$
$$F_C = (1.5 + W_2) = (1.5 + 2) = 3.5\ kN$$
$$F_A = 3.5\ kN$$

Bending moment diagram

The bending moment diagram is shown in Figure (c) and the values are tabulated here:

$$M_B = 0$$
$$M_C = -\,[1.5 \times 0.5] = -\,0.75\ kN.m$$
$$M_A = -\,[(1.5 \times 1.5) + (2 \times 1)] = -\,4.25\ kN.m$$



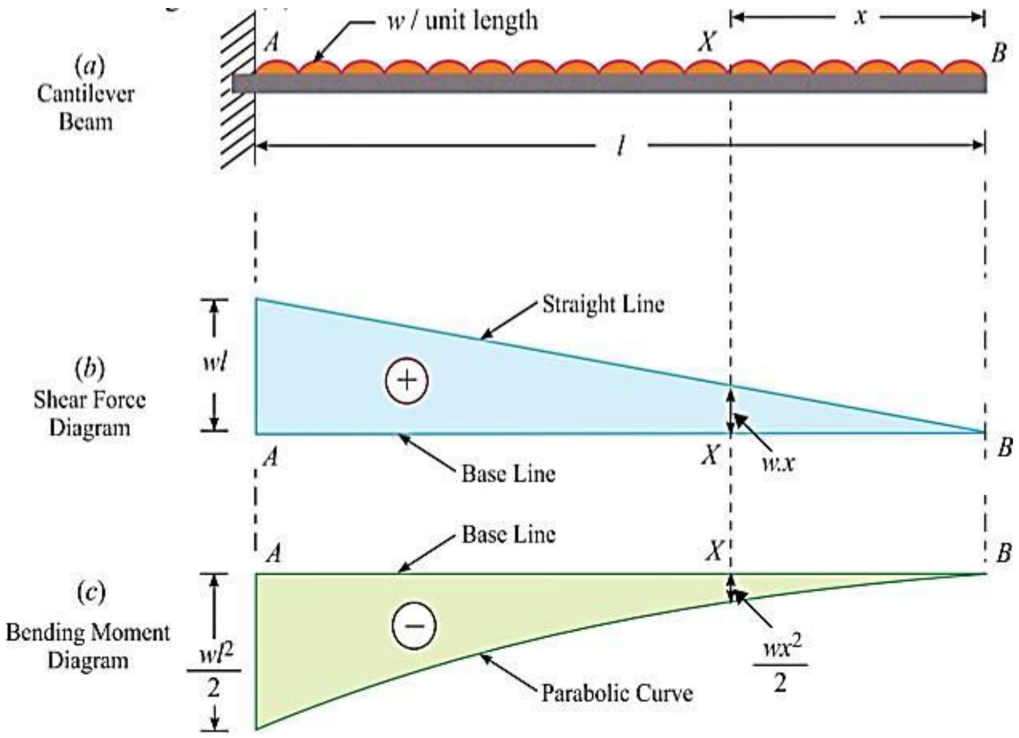
Cantilever with a Uniformly Distributed Load:

Consider a cantilever AB of length ***l*** and carrying a uniformly distributed load of w per unit length, over the entire length of the cantilever as shown in Figure.

We know that shear force at any section X, at a distance x from B,

$$F_x = W.x$$

Thus, we see that shear force is zero at B (where x = 0) and increases by a straight-line law to ***wl*** at A as shown in Figure.



We also know that bending moment at X,

$$M_x = -wx\,.\frac{x}{2} = \frac{-wx^2}{2}$$

$$\ldots\ldots (Minus\ sign\ due\ to\ hogging)$$

Thus, we also see that the bending moment is zero at B (where *x* = *0*) and increases in the form of a parabolic curve to $-\frac{wl^2}{2}$at B (where *x* = *l*) as shown in Figure above.

Example – 2: A cantilever beam AB, 2 m long carries a uniformly distributed load of 1.5 kN/m over a length of 1.6 m from the free end. Draw shear force and bending moment diagrams for the beam.

Solution. Given: span (*l*) = 2 m; Uniformly distributed load (w) = 1.5 kN/m and length of the cantilever CB carrying load (a) = 1.6 m.

Shear force diagram

The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_B = 0$$

$$F_C = w.a = 1.5 \times 1.6 = 2.4\ kN$$

$$F_A = 2.4\ kN$$

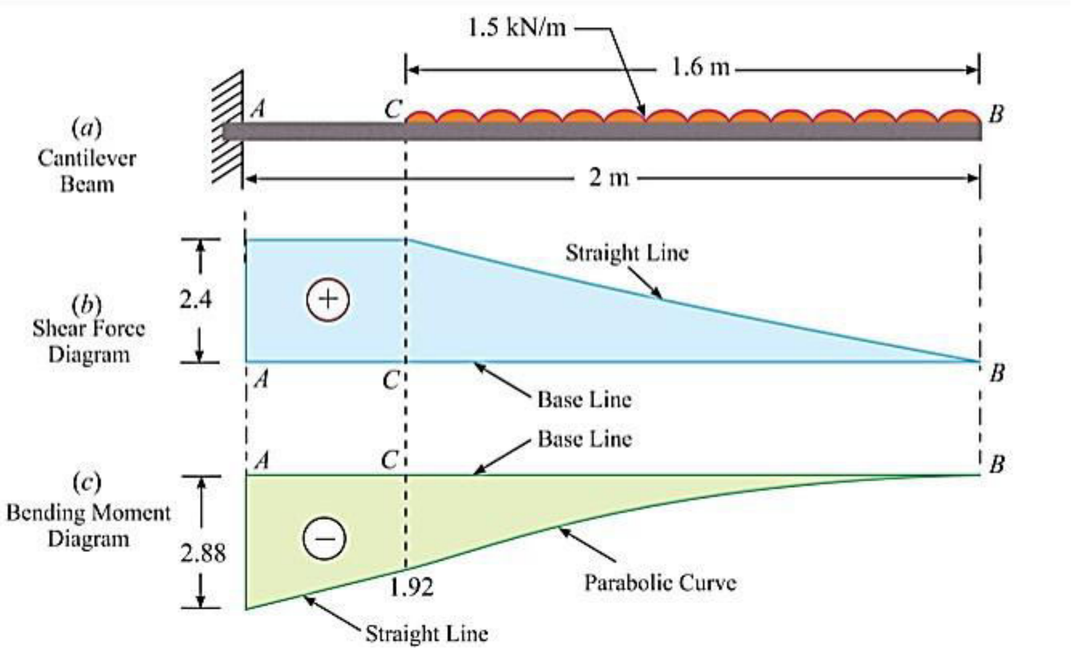
Bending moment diagram

The bending moment diagram is shown in Figure (c) and the values are tabulated here:

$$M_B = 0$$

$$M_C = -\frac{wa^2}{2} = -\frac{1.5 \times (1.6)^2}{2} = -1.92\ kN.m$$

$$M_A = -\left[(1.5 \times 1.6)\ (0.4 + \frac{1.6}{2})\right] = -2.88\ kN.m$$



Simply Supported Beam with a Point Load at its Mid-point:

Consider a simply supported beam AB of span l and carrying a point load W at its mid-point C as shown in the figure. Since the load is at the mid-point of the beam, therefore

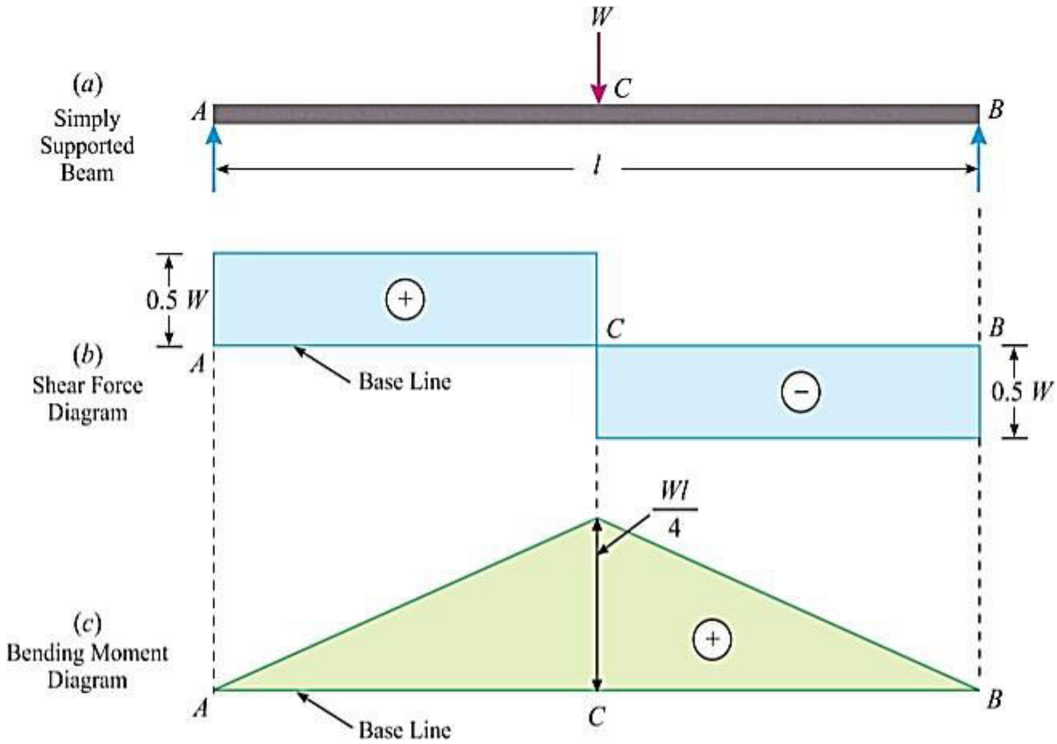
$$R_A + R_B = W$$

Taking moment about A, we get

$$R_B \times l = W \times \frac{l}{2}$$

$$\Rightarrow R_B = \frac{W}{2}$$

$$So\ R_A = \frac{W}{2}$$



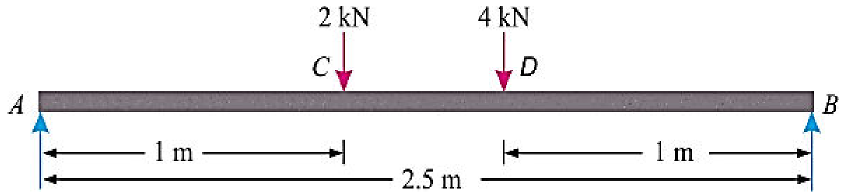
(Fig. simply supported beam with a point load)

Thus, we see that the shear force at any section between A and C (i.e., up to the point just before the load W) is constant and is equal to the unbalanced vertical force, i.e., + 0.5 W. Shear force at any section between C and B (i.e., just after the load W) is also constant and is equal to the unbalanced vertical force, i.e., – 0.5 W as shown in Figure (b).
We also see that the bending moment at A and B is zero. It increases by a straight-line law and is maximum at center of beam, where shear force changes sign as shown in Figure (c).
Therefore, bending moment at C,

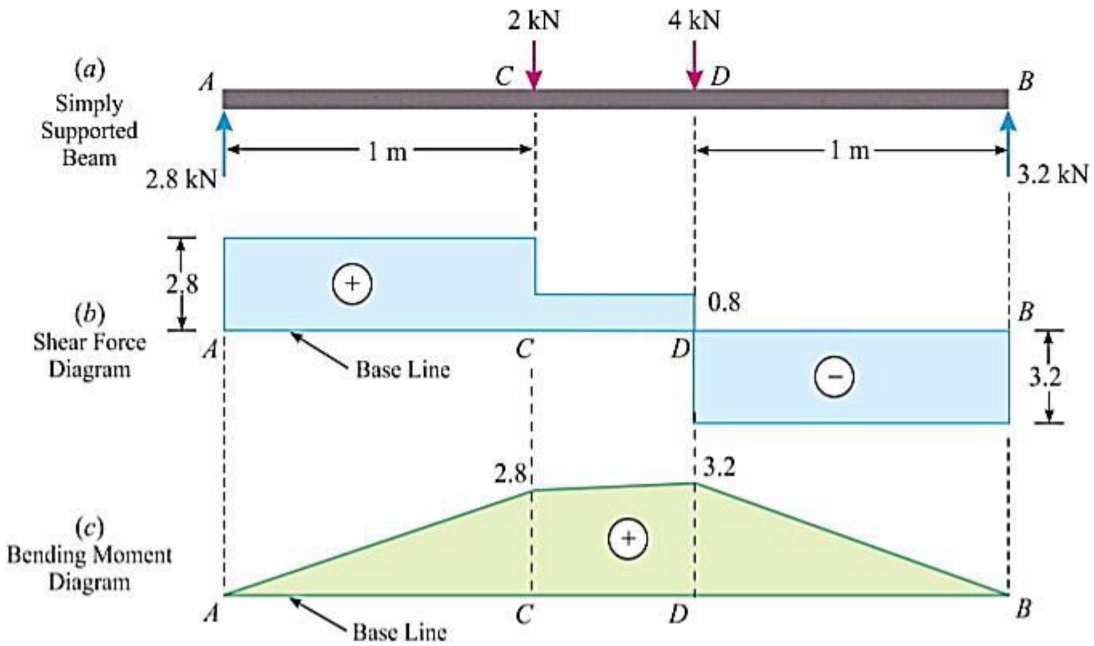
$$M_C = \frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4}$$

Note: B.M. at supports in case of a simply supported beam is always zero.

Example – 3: A simply supported beam AB of span 2.5 m is carrying two point loads as shown in the Figure. Draw the shear force and bending moment diagrams for the beam.



Solution. Given: Span (l) = 2.5 m; Point load at C (W_1) = 2 kN and point load at B (W_2) = 4 kN.



First of all, let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 2.5 = (2 \times 1) + (4 \times 1.5) = 8$$

$$R_B = 8/2.5 = 3.2 \text{ kN}$$

$$R_A = (2 + 4) - 3.2 = 2.8 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$R_A = + R_A = 2.8 \text{ kN}$$

$$F_C = + 2.8 - 2 = 0.8 \text{ kN}$$

$$F_D = 0.8 - 4 = - 3.2 \text{ kN}$$

$$F_B = - 3.2 \text{ kN}$$

Bending moment diagram
The bending moment diagram is shown in Figure (c) and the values are tabulated here:

$M_A = 0$

$M_C = 2.8 \times 1 = 2.8$ kN-m
$M_D = 3.2 \times 1 = 3.2$ kN-m

$M_B = 0$

<u>Simply Supported Beam with a Uniformly Distributed Load</u>:

Consider a simply supported beam AB of length l and carrying a uniformly distributed load of w per unit length as shown in Figure. Since the load is uniformly distributed over the entire length of the beam, therefore the reactions at the supports A,

$$R_A = R_B = \frac{wl}{2} = 0.5\,wl$$

We know that shear force at any section X at a distance x from A,

$$F_x = \frac{wl}{2} - wx = w\left(\frac{l}{2} - x\right)$$

$At\ A, x = 0, \qquad F_A = \frac{wl}{2}$

$At\ C, x = \frac{l}{2}, \qquad F_C = 0$

$At\ B, x = l, \qquad F_B = -\frac{wl}{2}$

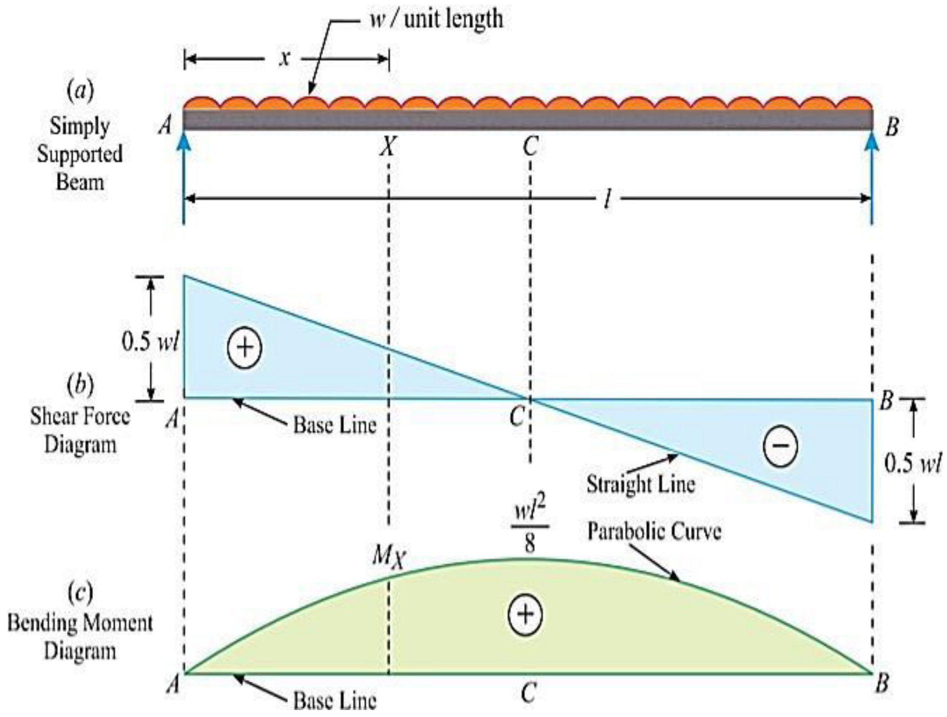
We also know that bending moment at any section at a distance x from A,

$$M_x = \frac{wl}{2}\,.\,x - wx\,.\,\frac{x}{2} = \frac{w}{2}(lx - x^2)$$

$At\ A, x = 0, \qquad M_A = 0$

$At\ B, x = l, \qquad M_B = 0$

$At\ C, x = \frac{l}{2}, \qquad M_C = \frac{w}{2}\left[l \times \frac{l}{2} - \left(\frac{l}{2}\right)^2\right] = \frac{w}{2}\left[\frac{l^2}{2} - \frac{l^2}{4}\right] = \frac{wl^2}{8}$



Example – 4: A simply supported beam 6 m long is carrying a uniformly distributed load of 5 kN/m over a length of 3 m from the right end. Draw the S.F. and B.M. diagrams for the beam and also calculate the maximum B.M. on the section.

Solution. Given: Span (l) = 6 m; Uniformly distributed load (w) = 5 kN/m and length of the beam CB carrying load (a) = 3 m

First of all, let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 6 = (5 \times 3) \times 4.5 = 67.5$$

$$\therefore R_B = 67.5/6 = 11.25 \text{ kN and}$$

$$R_A = (5 \times 3) - 11.25 = 3.75 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_A = +R_A = +3.75 \text{ kN}$$

$$F_C = +3.75 \text{ kN}$$

$$F_B = +3.75 - (5 \times 3) = -11.25 \text{ kN}$$

Bending moment diagram

The bending moment is shown in Figure (c) and the values are tabulated here:

$$M_A = 0$$

$$M_C = 3.75 \times 3 = 11.25 \text{ kN-m}$$

$$M_B = 0$$

We know that the maximum bending moment will occur at M, where the shear force changes sign. Let x be the distance between C and M. From the geometry of the figure between C and B, we find that

$$\frac{x}{3.75} = \frac{3-x}{11.25}$$

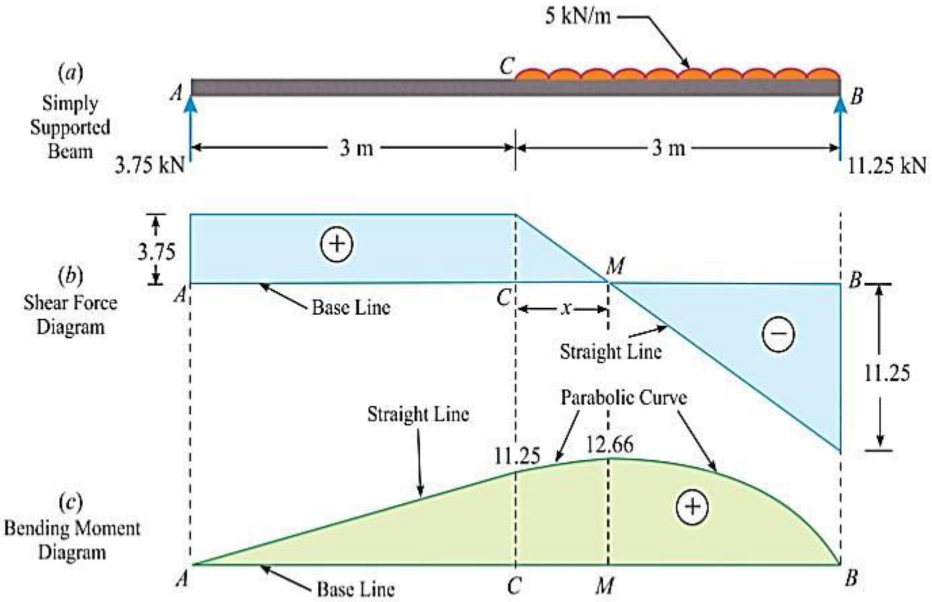
$$\Rightarrow 11.25\,x = 11.25 - 3.75x$$

$$\Rightarrow 15x = 11.25$$

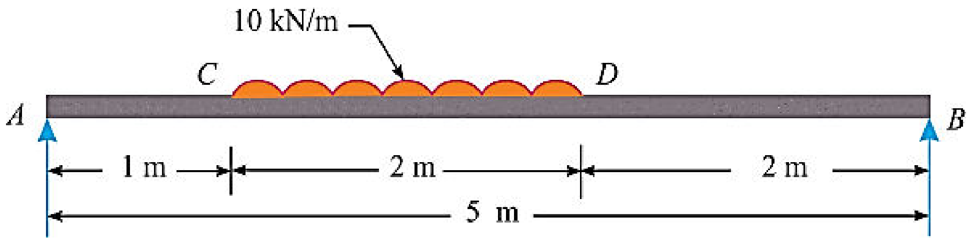
$$\Rightarrow x = \frac{11.25}{15} = 0.75\ m$$

So maximum bending moment at M,

$$M_M = 3.75 \times (3 + 0.75) - 5 \times 0.75 \times \frac{0.75}{2} = 12.66\ kN.m$$



Example – 5: A simply supported beam 5 m long is loaded with a uniformly distributed load of 10 kN/m over a length of 2 m as shown in Figure. Draw shear force and bending moment diagrams for the beam indicating the value of maximum bending moment.



Solution. Given: Span (l) = 5 m; Uniformly distributed load (w) = 10 kN/m and length of the beam CD carrying load (a) = 2 m.
First of all, let us find out the reactions R_A and R_B. Taking moments about A and equating the same,

$$R_B \times 5 = (10 \times 2) \times 2 = 40$$

$$\Rightarrow R_B = 40/5 = 8 \text{ kN}$$

$$\text{So, } R_A = (10 \times 2) - 8 = 12 \text{ kN}$$

Shear force diagram
The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_A = + RA = + 12 \text{ kN}$$

$$F_C = + 12 \text{ kN}$$

$$F_D = + 12 - (10 \times 2) = - 8 \text{ kN}$$

$$F_B = - 8 \text{ kN}$$

Bending moment diagram
The bending moment diagram is shown in Figure (c) and the values are tabulated here:

$$M_A = 0$$

$$M_C = 12 \times 1 = 12 \text{ kN-m}$$

$$M_D = 8 \times 2 = 16 \text{ kN-m}$$

$$M_B = 0$$

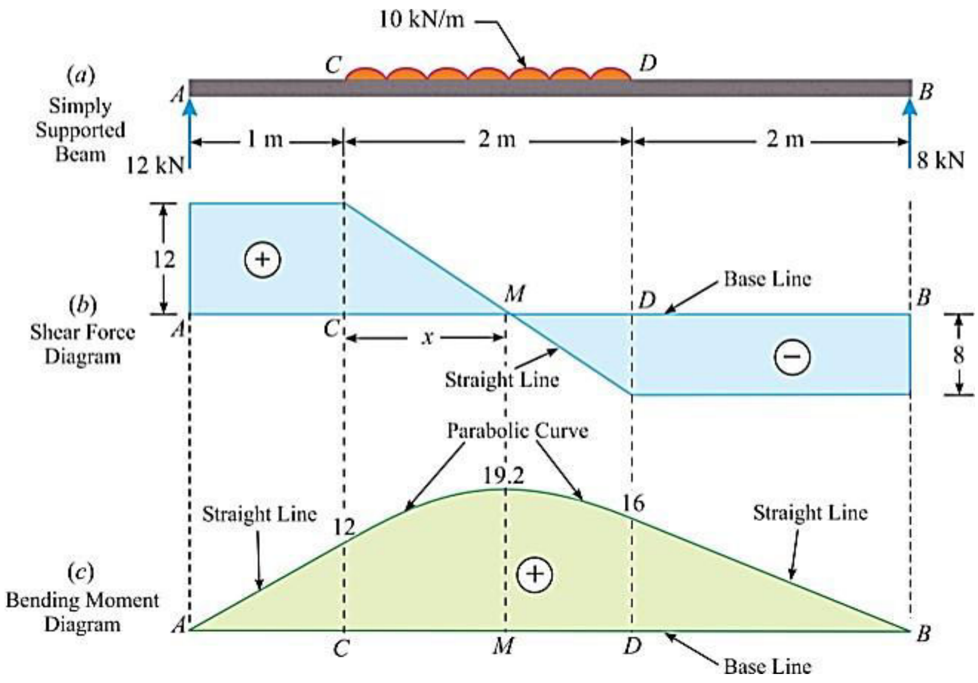
We know that maximum bending moment will occur at M, where the shear force changes sign. Let x be the distance between C and M. From the geometry of the figure between C and D, we find that

$$\frac{x}{12} = \frac{2-x}{8} \qquad or \qquad 8x = 24 - 12x$$

$$\Rightarrow 20\,x = 24 \qquad or \qquad x = 24/20 = 1.2\,m$$

So maximum bending moment at M,

$$M_M = 12(1+1.2) - (10 \times 1.2 \times \frac{1.2}{2}) = 19.2\ kN.m$$



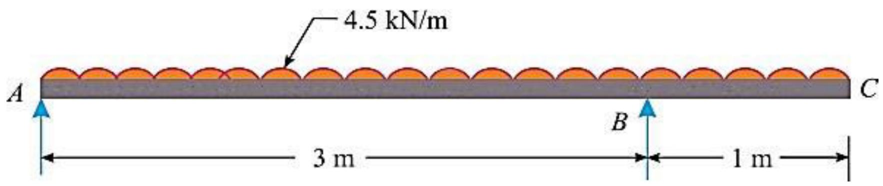
Overhanging Beam:

- It is a simply supported beam which overhangs (i.e., extends in the form of a cantilever) from its support.
- For the purposes of shear force and bending moment diagrams, the overhanging beam is analysed as a combination of a simply supported beam and a cantilever. An overhanging beam may overhang on one side only or on both sides of the supports.

Point of Contraflexure:

- It is the point at which the bending moment changes sign (i.e., from + ve to – ve or vice versa)
- In this point the value of bending moment is zero.
- At this point the beam flexes in opposite direction. This point is also called as point of inflection.

Example – 6: An overhanging beam ABC is loaded as shown in Figure. Draw the shear force and bending moment diagrams and find the point of contraflexure, if any.



Solution. Given: Span (l) = 4 m; Uniformly distributed load (w) = 4.5 kN/m and overhanging length (c)=1m.

First of all, let us find out the reactions R_A and R_B. Taking moment about A and equating the same,

$$R_B \times 3 = (4.5 \times 4) \times 2 = 36$$

$$\Rightarrow R_B = 36/3 = 12 \text{ kN}$$

$$\text{So,} \quad R_A = (4.5 \times 4) - 12 = 6 \text{ kN}$$

Shear force diagram

The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_A = + R_A = + 6 \text{ kN}$$

$$F_B = + 6 - (4.5 \times 3) + 12 = 4.5 \text{ kN}$$

$$F_{A-B} = +6 - (4.5 \times 3) = - 7.5 \text{ kN}$$

$$F_C = + 4.5 - (4.5 \times 1) = 0$$

Bending moment diagram
The bending moment diagram is shown in Figure (c) and the values are tabulated here:

$$M_A = 0$$

$$M_B = (6 \times 3) - (4.5 \times 3 \times 1.5) = -\,2.25\ kN.m$$

$$M_C = 0$$

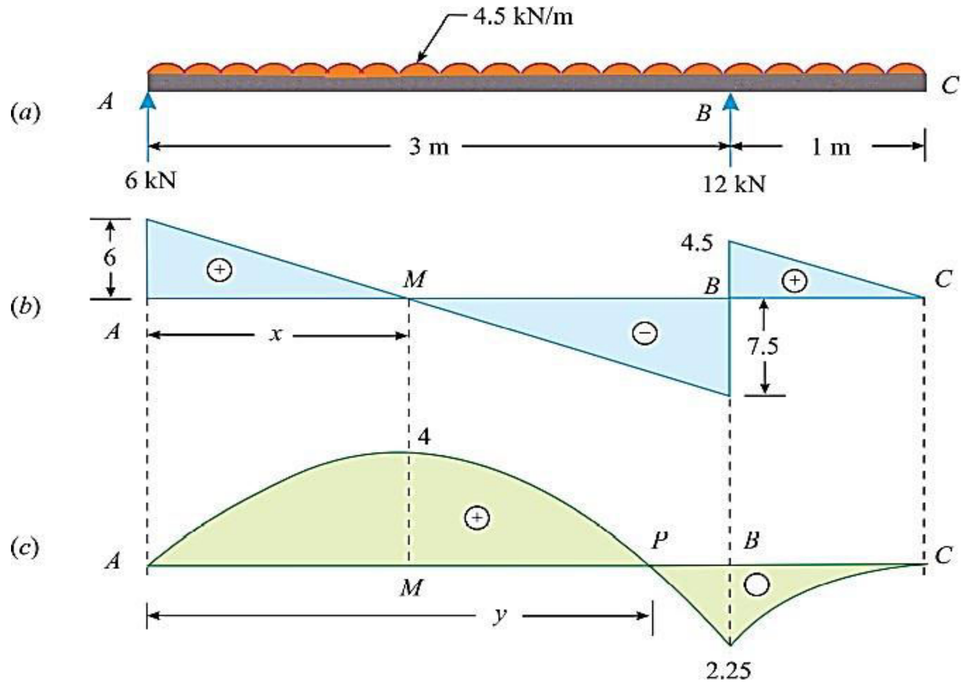
We know that maximum bending moment will occur at M, where the shear force changes sign. Let *x* be the distance between A and M. From the geometry of the figure between A and B, we find that

$$\frac{x}{6} = \frac{3-x}{7.5} \qquad or \qquad 7.5x = 18 - 6x$$

$$\Rightarrow 13.5\,x = 18 \qquad or \qquad x = 18/13.5 = 1.33\ m$$

So maximum bending moment at M,

$$M_M = (6 \times 1.33) - (4.5 \times 1.33 \times \frac{1.33}{2}) = 4\ kN.m$$



Point of contraflexure
Let P be the point of contraflexure at a distance y from the support A. We know that bending moment at P

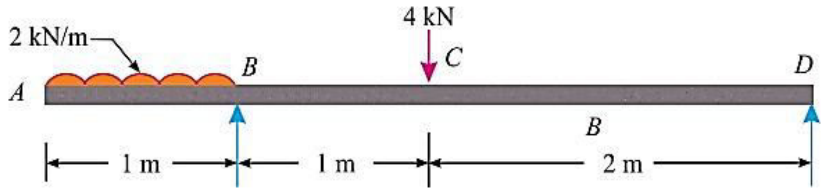
$$M_P = (6 \times y) - (4.5 \times y \times \frac{y}{2}) = 0$$

$$\Rightarrow 6y - 2.25y^2 = 0$$

$$\Rightarrow 2.25y^2 = 6y$$

$$\Rightarrow y = \frac{6}{2.25} = 2.67\ m$$

Example – 7: A beam ABCD, 4 m long is overhanging by 1 m and carries load as shown in Figure. Draw the shear force and bending moment diagrams for the beam and locate the point of contraflexure.



Solution. Given: Span (l) = 4 m; Uniformly distributed load over AB (w) = 2 kN/m and point load at C (W) = 4 kN.
First of all, let us find out the reactions R_B and R_D. Taking moments about B and equating the same,

$$R_D \times 3 = (4 \times 1) - (2 \times 1) \times 1/2 = 3$$

$$\Rightarrow R_D = 3/3 = 1 \text{ kN}$$

$$\text{So,} \quad R_B = (2 \times 1) + 4 - 1 = 5 \text{ kN}$$

Shear force diagram
The shear force diagram is shown in Figure (b) and the values are tabulated here:

$$F_A = 0$$

$$F_B = 0 - (2 \times 1) + 5 = +3 \text{ kN}$$

$$F_{A\text{-}B} = -(2 \times 1) = -2 \text{ kN}$$

$$F_C = +3 - 4 = -1 \text{ kN}$$

$$F_D = -1 \text{ kN}$$

Bending moment diagram
The bending moment diagram is shown in Figure (c) and the values are tabulated here:

$$M_A = 0$$

$$M_B = -(2 \times 1)\, 0.5 = -1 \text{ kN-m}$$

$$M_C = 1 \times 2 = +2 \text{ kN-m}$$

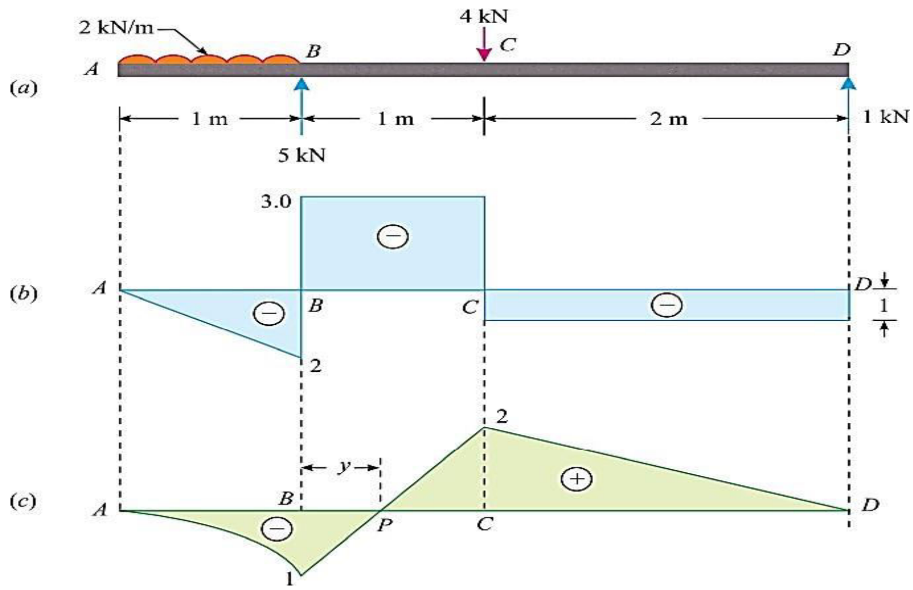
$$M_D = 0$$

We know that maximum bending moment occurs either at B or C, where the shear force changes sign. From the geometry of the bending moment diagram, we find that maximum negative bending moment occurs at B and maximum positive bending moment occurs at C.

Point of contraflexure
Let P be the point of contraflexure at a distance y from the support B. From the geometry of the figure between B and C, we find that

$$\frac{y}{1} = \frac{1-y}{2} \quad or \quad 2y = 1 - y$$

$$\Rightarrow 3y = 1 \quad or \quad y = 1/3 = 0.33\, m$$



<u>Possible Short Type Question with Answers</u>:

1. Define shear force and bending moment. (W – 2019)

- The shear force (briefly written as S.F.) at the cross-section of a beam may be defined as the unbalanced vertical force to the right or left of the section.
- The bending moment (briefly written as B.M.) at the cross-section of a beam may be defined as the algebraic sum of the moments of the forces, to the right or left of the section.

2. Define point of contraflexure. (W – 2019, 2020)

- It is the point at which the bending moment changes sign (i.e., from + ve to – ve or vice versa)
- In this point the value of bending moment is zero.
- At this point the beam flexes in opposite direction. This point is also called as point of inflection.

3. Define cantilever beam with example. (W – 2020)

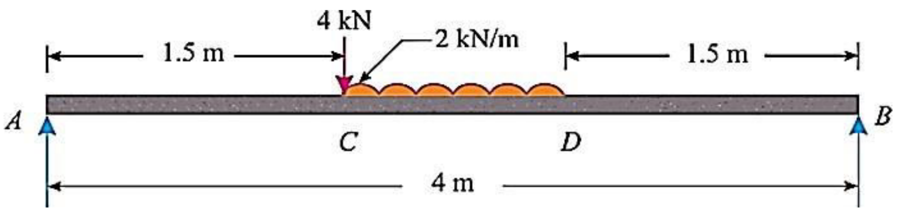
- A beam fixed at one end and free (unsupported) at the other end is called a cantilever beam or simply cantilever.
- Examples: A good example is a balcony, it is supported at one end only, the rest of the beams extends over the open space. Other examples are a cantilever roof in a bus shelter, car park, cantilever bridge (the forth bridge in Scotland is an example of a cantilever truss bridge).

<u>Possible Long Type Questions</u>:

1. A simply supported beam of 6 m span carries a point load of 50 kN at a distance of 5 m from its left end. Draw S.F & B.M diagram for the beam. **(W - 2019)**

Hints: Refer examples of S.S.B carrying point loads

2. A simply supported beam of 4 m span is carrying loads as shown in the figure. Draw the shear force and bending moment diagrams for the beam. **(W - 2019)**



Hints: Refer examples of S.S.B carrying point loads and u.d.l

3. Show diagrammatically different types of beams and loads. **(W - 2020)**

Hints: Refer article 4.1 (page no 01 & 02)

4. A simply supported beam of length 6 m carries point loads of 3 kN and 6 kN at distances of 2 m and 4 m from the left end. Draw the shear force and bending moment diagram for the beam. **(W - 2020).**

Hints: Refer examples of S.S.B carrying point loads

CHAPTER 05

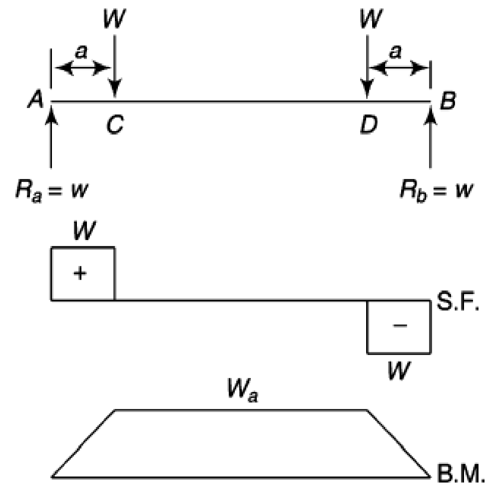
THEORY OF SIMPLE BENDING

LEARNING OBJECTIVES:

Assumptions in the theory of bending
Bending equation, Moment of resistance, Section modulus& neutral axis.
Solve simple problems.

Introduction:

- The bending moment at a section tends to bend or deflect the beam and the internal stresses resist its bending. The process of bending stops, when every cross-section sets up full resistance to the bending moment. The resistance, offered by the internal stresses, to the bending, is called bending stress, and the relevant theory is called the theory of simple bending.
- **Pure bending or simple bending**: If a member is subjected to equal and opposite couples acting in the same longitudinal plane, the member is said to be in pure bending. A beam or a part of it is said to be in a state of pure bending when it bends under the action of constant bending moment, without any shear force (i.e., S.F = 0 & B.M = constant).



Theory of simple bending:

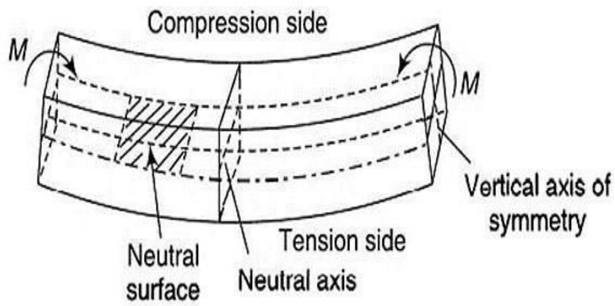
- Whenever a beam is subjected to simple bending (pure bending) the fibres on one side of beam are subjected to compression while the fibres on the other side are subjected to tension, in between the top and bottom fibres there is a surface where bending stress is zero, this is called neutral surface (neutral layer).

Neutral layer:

- It is a layer in which longitudinal fibres do not change in length. At this layer stress and strain are zero. On one side of layer longitudinal fibres will elongate and on the other side, longitudinal fibres will contract.

Neutral axis:

- It is the line of intersection of neutral layer with the cross section of plane. At neutral axis, stress and strain are zero.



Assumptions in the Theory of Bending:

The following assumptions are made in the theory of simple bending:

1. The material of the beam is perfectly homogeneous (i.e., of the same kind throughout) and isotropic (i.e., of equal elastic properties in all directions).
2. The beam material is stressed within its elastic limit and thus, obeys Hooke's law.
3. The transverse sections, which were plane before bending, remains plane after bending also.
4. Each layer of the beam is free to expand or contract, independently, of the layer above or below it.
5. The value of E (Young's modulus of elasticity) is the same in tension and compression.
6. The beam is in equilibrium i.e., there is no resultant pull or push in the beam section

Bending equation, Moment of resistance, Section modulus& neutral axis:

Bending equation:

Consider a small length dx of a beam subjected to a bending moment as shown in Figure (a). As a result of this moment, let this small length of beam bend into an arc of a circle with O as centre as shown in Fig. (b).

Let M = Moment acting at the beam

θ = Angle subtended at the centre by the arc

R = Radius of curvature of the beam.

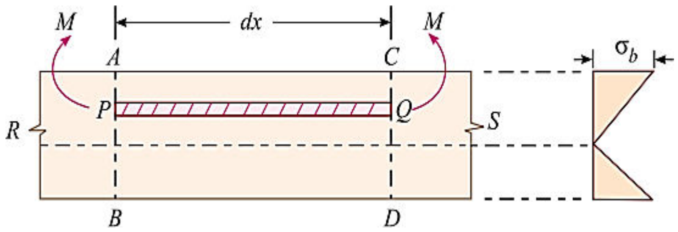


Figure (a)

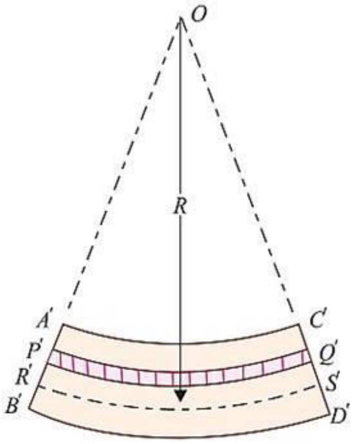


Figure (b)

Now consider a layer PQ at a distance y from RS (the neutral axis of the beam). Let this layer be compressed to $P'Q'$ after bending as shown in Figure (b).

Let $R'S'$ subtend an angle θ at the centre of curvature.

$$\therefore R'S' = R\theta \; and \; P'Q' = (R - y)\theta$$

Initially the parallel layers would have equal lengths, So that RS = PQ and since there is no stress at the neutral axis, then there is no strain.

So, $RS = R'S' = PQ$

$$Now \; the \; strain \; in \; PQ = \frac{PQ - P'Q'}{PQ}$$

$$\Rightarrow Strain = \frac{R'S' - P'Q'}{PQ}$$

$$\Rightarrow Strain = \frac{R\theta - (R - y)\theta}{R\theta} = \frac{y}{R}$$

Now if the stress in $PQ = \sigma$ and Young's modulus is E, then

$$Strain = \frac{\sigma}{E}$$

$$\therefore \frac{\sigma}{E} = \frac{y}{R} \; or \; \frac{\sigma}{y} = \frac{E}{R} \ldots\ldots\ldots\ldots\ldots (i)$$

Consider a section of the beam as shown in Figure Let NA be the neutral axis of the section. Now consider a small layer PQ of the beam section at a distance y from the neutral axis as shown in Figure.

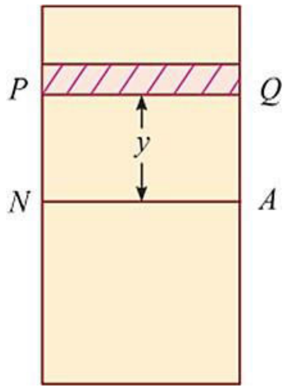
Let, δa = Area of the layer PQ

Then the normal force on this area (δa)

$$= \sigma.\delta a = \frac{E}{R}.y.\delta a$$

Now moment of this force about the neutral axis is

$$= \frac{E}{R} \cdot y . \delta a \times y = \frac{E}{R} \cdot y^2 . \delta a$$



The algebraic sum of all such moments about the neutral axis must be equal to M.

$$Therefore, M = \sum \frac{E}{R} \cdot y^2 . \delta a = \frac{E}{R} \sum y^2 . \delta a$$

The expression $\boldsymbol{\Sigma\, y^2 .\, \delta a}$ represents the moment of inertia of the area of the whole section about the neutral axis. Therefore

$$M = \frac{E}{R} \times I \qquad or \qquad \frac{M}{I} = \frac{E}{R} \ldots\ldots\ldots\ldots (ii)$$

So, from equation (i) and (ii) we get,

$$\boldsymbol{\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}}$$

Where, M = Moment of resistance

I = Moment of inertia of the section about neutral axis

E = Young's modulus of elasticity

R = Radius of curvature of N.A.

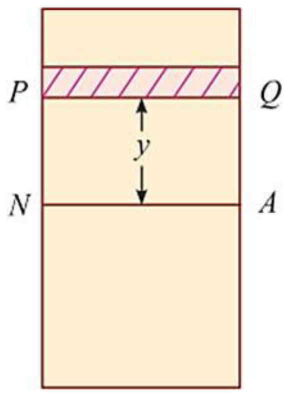
σ = Bending stress

The above equation is known as the **'Bending equation'**.

<u>Position of Neutral Axis</u>:

Consider a section of the beam as shown in Figure Let NA be the neutral axis of the section. Now consider a small layer PQ of the beam section at a distance y from the neutral axis as shown in Figure.

Let, δa = Area of the layer PQ



Then the normal force on this area (δa)

$$= \sigma . \delta a = \frac{E}{R} \cdot y . \delta a$$

Net normal force on the cross section

$$= \frac{E}{R} \sum y \,.\, \delta a$$

For pure bending, net normal force on the cross section = 0

$$\Rightarrow \frac{E}{R} \sum y \,.\, \delta a = 0$$

$$\Rightarrow \sum y \,.\, \delta a = 0$$

Now $\sum y \,.\, \delta a$ is the moment of the sectional area about the neutral axis, and since this moment is zero, the axis must pass through the centre of area.

Hence the neutral axis or neutral layer, passes through the centre of area or centroid.

<u>Section modulus</u>:

- From bending equation, we have

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\Rightarrow \sigma = \frac{My}{I} = \frac{M}{I/y}$$

$$\Rightarrow \sigma = \frac{M}{Z}$$

$Where, Z = Section\ modulus = \frac{I}{y}$

Definition of Z:

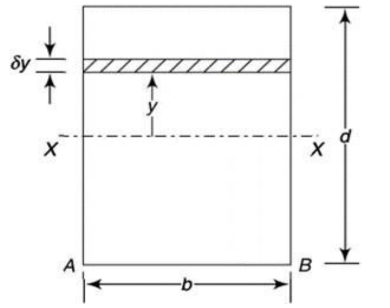
- It is the ratio of moment of inertia of the beam cross section about the neutral axis (I) to the distance of farthest point(y_{max}) of the section from the neutral axis.

$$\boldsymbol{Z = \frac{I}{y_{max}}}$$

Section moduli of Rectangular & circular section:

Rectangular section:

- We know that moment of inertia of a rectangular section about an axis through its centre of gravity,

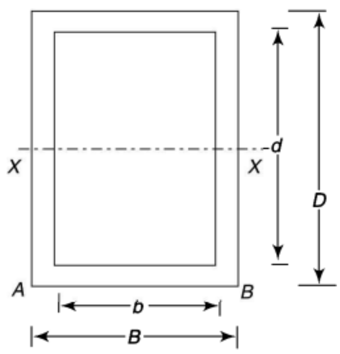


$$I = \frac{bd^3}{12} \quad and \quad y_{max} = \frac{d}{2}$$

$$\therefore \boldsymbol{Z = \frac{I}{y_{max}} = \frac{bd^3/12}{d/2} = \frac{bd^2}{6}}$$

Hollow rectangular section:

- We know that moment of inertia of a hollow rectangular section about an axis through its centre of gravity,

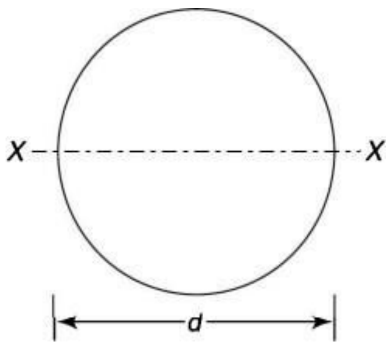


$$I = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{BD^3 - bd^3}{12} \quad \& \quad y_{max} = \frac{D}{2}$$

$$\therefore \boldsymbol{Z = \frac{I}{y_{max}} = \frac{(BD^3 - bd^3)/12}{D/2} = \frac{BD^3 - bd^3}{6D}}$$

Circular section:

- We know that moment of inertia of a circular section about an axis through its centre of gravity,

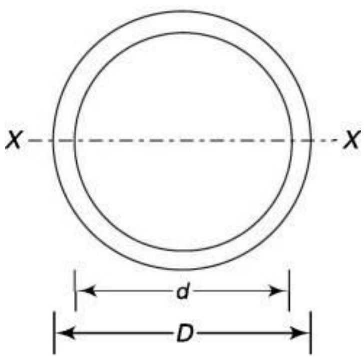


$$I = \frac{\pi d^4}{64} \quad and \quad y_{max} = \frac{d}{2}$$

$$\therefore \boldsymbol{Z = \frac{I}{y_{max}} = \frac{\pi d^4/64}{d/2} = \frac{\pi d^3}{32}}$$

Hollow circular section:

- We know that moment of inertia of a hollow circular section about an axis through its centre of gravity,



$$I = \frac{\pi(D^4 - d^4)}{64} \, and \quad y_{max} = \frac{D}{2}$$

$$\therefore \boldsymbol{Z = \frac{I}{y_{max}} = \frac{\pi(D^4 - d^4)/64}{D/2} = \frac{\pi}{32}\left(\frac{D^4 - d^4}{D}\right)}$$

Moment of resistance:

- The maximum bending moment which can be carried by a given section for a given maximum value of stress is known as the moment of resistance.

Or

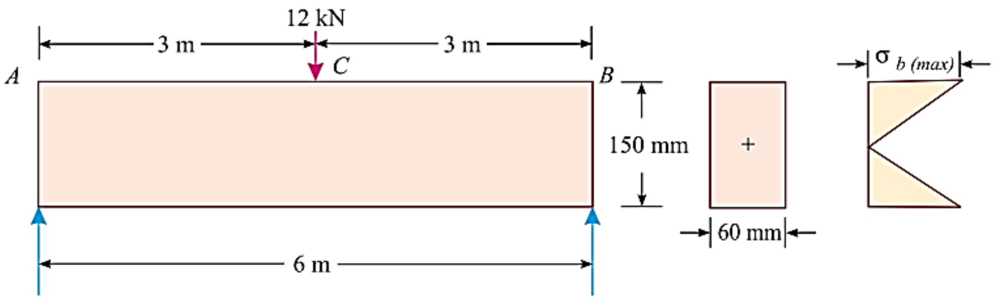
- It is the product of maximum bending stress (σ) and section modulus (Z).
- Mathematically,

$$\boldsymbol{M = \sigma \times Z}$$

Solve simple problems:

Example – 1: A rectangular beam 60 mm wide and 150 mm deep is simply supported over a span of 6 m. If the beam is subjected to central point load of 12 kN, find the maximum bending stress induced in the beam section.

Solution. Given: Width (b) = 60 mm; Depth (d) = 150 mm; Span (l) = 6×10^3 mm and load (W) = 12 kN = 12×10^3 N



We know that maximum bending moment at the centre of a simply supported beam subjected to a central point load,

$$M = \frac{Wl}{4} = \frac{(12 \times 10^3) \times (6 \times 10^3)}{4} = 18 \times 10^6 \ N.mm$$

and section modulus of the rectangular section,

$$Z = \frac{bd^2}{6} = \frac{60 \times (150)^2}{6} = 225 \times 10^3 \ mm^3$$

Maximum bending stress,

$$\sigma = \frac{M}{Z} = \frac{18 \times 10^6}{225 \times 10^3} = 80 \ \ N/mm^2 = 80 \ MPa$$

Example – 2: A rectangular beam 300 mm deep is simply supported over a span of 4 metres. What uniformly distributed load the beam may carry, if the bending stress is not to exceed 120 MPa. Take I = 225×10^6 mm^4.

Solution. Given: Depth (d) = 300 mm; Span (l) = 4 m = 4×10^3 mm; Maximum bending stress (σ_{max}) = 120 MPa = 120 N/mm^2 and moment of inertia of the beam section (I) = 225×10^6 mm^4

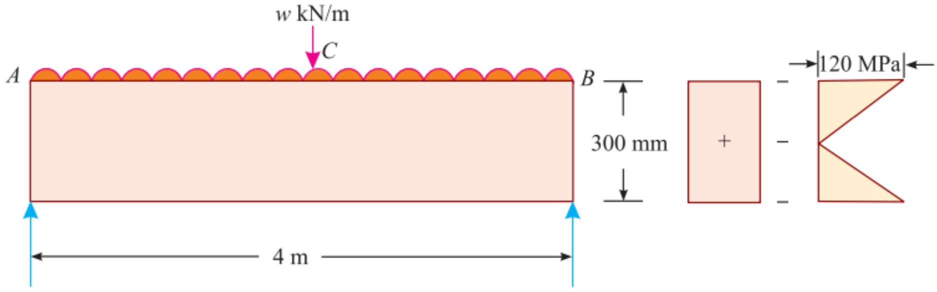
Let, w = Uniformly distributed load the beam can carry

We know that distance between the neutral axis of the section and extreme fibre,

$$y = \frac{d}{2} = \frac{300}{2} = 150 \ mm$$

and section modulus of the rectangular section,

$$Z = \frac{I}{y} = \frac{225 \times 10^6}{150} = 1.5 \times 10^6 \ mm^3$$



Moment of resistance,

$$M = \sigma_{max} \times Z = 120 \times (1.5 \times 10^6) = 180 \times 10^6\ N.mm$$

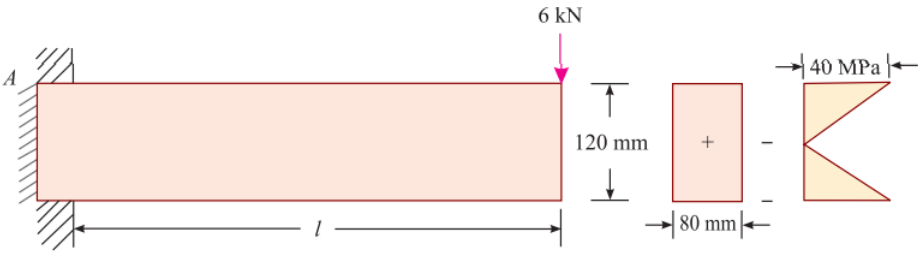
We also know that maximum bending moment at the centre of a simply supported beam subjected to a uniformly distributed load (M),

$$180 \times 10^6 = \frac{wl^2}{8} = \frac{w \times (4 \times 10^3)^2}{8} = 2 \times 10^6\ w$$

$$\Rightarrow w = \frac{180}{2} = 90\ N/mm = 90\ kN/m$$

Example – 3: A cantilever beam is rectangular in section having 80 mm width and 120 mm depth. If the cantilever is subjected to a point load of 6 kN at the free end and the bending stress is not to exceed 40 MPa, find the span of the cantilever beam.

Solution. Given: Width (b) = 80 mm; Depth (d) = 120 mm; Point load (W) = 6 kN= 6×10^3 N and maximum bending stress (σ_{max}) = 40 MPa = 40 N/mm^2.



Let, l = Span of the cantilever beam

We know that section modulus of the rectangular section,

$$Z = \frac{bd^2}{6} = \frac{80 \times (120)^2}{6} = 192 \times 10^3\ mm^3$$

and maximum bending moment at the fixed end of the cantilever subjected to a point load at the free end,

$$M = Wl = (6 \times 10^3) \times l$$

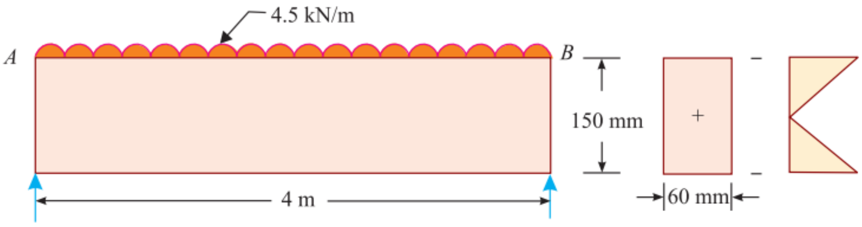
Maximum bending stress (σ_{max}),

$$40 = \frac{M}{Z} = \frac{(6 \times 10^3) \times l}{192 \times 10^3} = \frac{l}{32}$$

$$\Rightarrow l = 40 \times 32 = 1280\ mm = 1.28\ m$$

Example – 4: A rectangular beam 60 mm wide and 150 mm deep is simply supported over a span of 4 metres. If the beam is subjected to a uniformly distributed load of 4.5 kN/m, find the maximum bending stress induced in the beam.

Solution. Given: Width (b) = 60 mm; Depth (d) = 150 mm; Span (l) = 4 m = 4×10^3 mm and uniformly distributed load (w) = 4.5 kN/m = 4.5 N/mm.



We know that section modulus of the rectangular section,

$$Z = \frac{bd^2}{6} = \frac{60 \times (150)^2}{6} = 225 \times 10^3 mm^3$$

and maximum bending moment at the centre of a simply supported beam subjected to a uniformly distributed load,

$$M = \frac{wl^2}{8} = \frac{4.5 \times (4 \times 10^3)^2}{8} = 9 \times 10^6 \ N.mm$$

Maximum bending stress,

$$\sigma_{max} = \frac{M}{Z} = \frac{9 \times 10^6}{225 \times 10^3} = 40 \ \ N/mm^2 = 40 \ MPa$$

Possible Short Type Question with Answers:

01. Define section modulus. (W – 2019, 2020)

- It is the ratio of moment of inertia of the beam cross section about the neutral axis (I) to the distance of farthest point(y_{max}) of the section from the neutral axis.

$$\boldsymbol{Z = \frac{I}{y_{max}}}$$

02. Define moment of resistance.

- The maximum bending moment which can be carried by a given section for a given maximum value of stress is known as the moment of resistance.

Or

- It is the product of maximum bending stress (σ) and section modulus (Z).

- Mathematically,

$$\boldsymbol{M = \sigma \times Z}$$

03. what is pure bending?

- If a member is subjected to equal and opposite couples acting in the same longitudinal plane, the member is said to be in pure bending

Possible Long Type Questions:

01. State the assumptions made in theory of simple bending. **(W – 2019)**
Hints: Refer article 5.1 (page no 01)

02. A beam 3 m long has rectangular section of 80 mm width and 120 mm depth. If the beam is carrying a uniformly distributed load of 10 kN/m, find the maximum bending stress developed in the beam. **(W – 2019)**
Hints: Refer Example - 4 (page no 06)

03. Derive the formula of section modulus for rectangular section and circular section. **(W – 2020)**
Hints: Refer (page no 04)

04. Prove the relation

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

Where, M = Bending Moment
I = Moment of inertia
E = Young's modulus
R = Radius of curvature
σ = Bending stress in a fibre, at a distance y from the neutral axis **(W – 2020)**

Hints: Refer article 5.2 (page no 02)

CHAPTER 06

COMBINED DIRECT & BENDING STRESSES

LEARNING OBJECTIVES:

Define column
Axial load, Eccentric load on column
Direct stresses, Bending stresses, Maximum& Minimum stresses. Numerical problems on above.
Buckling load computation using Euler's formula (no derivation) in Columns with various end conditions.

Define column:

- A column is a long vertical slender bar or vertical member, subjected to an axial compressive load and fixed rigidly at both ends.

Axial load, Eccentric load on column:

Axial load:

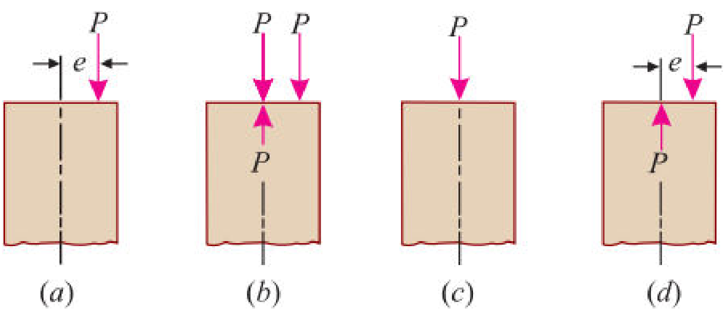
- A load, whose line of action coincide with the axis of a column, is known as an axial load.

Eccentric load

- A load, whose line of action does not coincide with the axis of a column or a strut, is known as an eccentric load.

Example: A bucket full of water, carried by a person in his hand, is an excellent example of an eccentric load. A little consideration will show that the man will feel this load as more severe than the same load, if he had carried the same bucket over his head. The simple reason for the same is that if he carries the bucket in his hand, then in addition to his carrying bucket, he has also to lean or bend on the other side of the bucket, so as to counteract any possibility of his falling towards the bucket. Thus, we say that he is subjected to:
1. Direct load, due to the weight of bucket (including water) and
2. Moment due to eccentricity of the load

Columns with Eccentric Loading:



Consider a column subjected to an eccentric loading. The eccentric load may be easily analysed as shown in Figure above and as discussed below:

- The given load P, acting at an eccentricity of e, is shown in Fig. (a).
- Let us introduce, along the axis of the strut, two equal and opposite forces P as shown in Fig. (b).
- The forces thus acting, may be split up into three forces.
- One of these forces will be acting along the axis of the strut. This force will cause a direct stress as shown in Fig. (c).
- The other two forces will form a couple as shown in Fig. (d). The moment of this couple will be equal to P × e (This couple will cause a bending stress).

Direct stresses, Bending stresses, Maximum& Minimum stresses. Numerical problems on above:

Symmetrical Columns with Eccentric Loading about One Axis:

Consider a column ABCD subjected to an eccentric load about one axis (i.e., about y-y axis) as shown in the figure.

Let, P = Load acting on the column,
 e = Eccentricity of the load,
 b = Width of the column section and
 d = Thickness of the column.

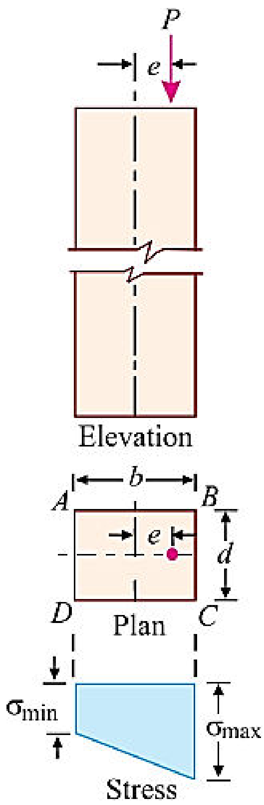
Area of column section, $\boldsymbol{A = b \cdot d}$

Moment of inertia of the column section about an axis through its centre of gravity and parallel to the axis about which the load is eccentric (i.e., y-y axis in this case),

$$I = \frac{db^3}{12}$$

and modulus of section,

$$Z = \frac{I}{y} = \frac{db^3/12}{b/2} = \frac{db^2}{6}$$



We know that direct stress on the column due to the load,

$$\sigma_0 = \frac{P}{A}$$

and moment due to load,

$$M = P.e$$

Bending stress at any point of the column section at a distance y from y-y axis,

$$\sigma_b = \frac{M.y}{I} = \frac{M}{Z}$$

Now for the bending stress at the extreme, let us substitute y = b/2 in the above equation,

$$\sigma_b = \frac{M.(b/2)}{db^3/12} = \frac{6M}{db^2} = \frac{6\ P.e}{A.b}$$

We know that an eccentric load causes a direct stress as well as bending stress. It is thus obvious that the total stress at the extreme fibre,

$$\sigma = \sigma_0\ \pm\ \sigma_b = \frac{P}{A} \pm \frac{6\ P.e}{A.b} \ldots (In\ terms\ of\ eccentricity)$$

$$= \frac{P}{A} \pm \frac{M}{Z} \ldots (In\ terms\ of\ modulus\ of\ section)$$

The +ve or –ve sign will depend upon the position of the fibre with respect to the eccentric load. A little consideration will show that the stress will be maximum at the corners B and C (because these corners are near the load), whereas the stress will be minimum at the corners A and D (because these corners are away from the load). The total stress along the width of the column will vary by a straight-line law.

The maximum stress,

$$\sigma_{max} = \frac{P}{A} + \frac{6\ P.e}{A.b} = \frac{P}{A}(1 + \frac{6e}{b}) \ldots\ldots\ldots (In\ terms\ of\ eccentricity)$$

$$= \frac{P}{A} + \frac{M}{Z} \ldots\ldots\ldots (In\ terms\ of\ modulus\ of\ section)$$

The minimum stress,

$$\sigma_{min} = \frac{P}{A} - \frac{6\,P.e}{A.b} = \frac{P}{A}(1 + \frac{6e}{b}) \dots\dots\dots (In\ terms\ of\ eccentricity)$$

$$= \frac{P}{A} - \frac{M}{Z} \dots\dots\dots\ \dots\dots (In\ terms\ of\ modulus\ of\ section)$$

Notes: From the above equations, we find that

- If σ_0 is greater than σ_b, the stress throughout the section, will be of the same nature (i.e., compressive).
- If σ_0 is equal to σ_b, even then the stress throughout the section will be of the same nature. The minimum stress will be equal to zero, whereas the maximum stress will be equal to $2 \times \sigma_0$.
- If σ_0 is less than σ_b, then the stress will change its sign (partly compressive and partly tensile).

Example – 1: A rectangular strut is 150 mm and 120 mm thick. It carries a load of 180 kN at an eccentricity of 10 mm in a plane bisecting the thickness. Find the maximum and minimum intensities of stress in the section.

Solution. Given: Width (b) = 150 mm ; Thickness (d) = 120 mm ; Load (P) = 180 kN = 180×10^3 N and eccentricity (e) = 10 mm.

Maximum intensity of stress in the section

We know that area of the strut,

$$A = b \times d = 150 \times 120 = 18\,000 \text{ mm}^2$$

and maximum intensity of stress in the section,

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{6e}{b}\right) = \frac{1800 \times 10^3}{18\,000}\left(1 + \frac{6 \times 10}{150}\right) \text{ N/mm}^2$$

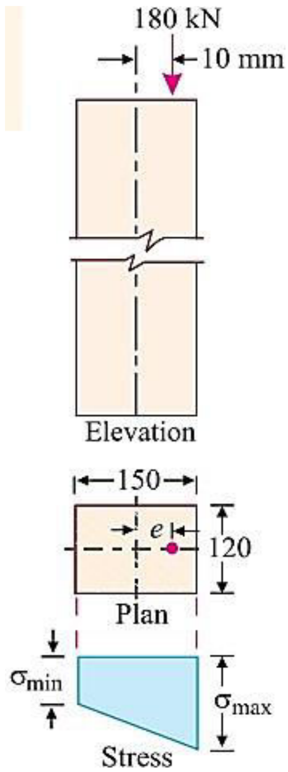
$$= 10\,(1 + 0.4) = 14 \text{ N/mm}^2 = \mathbf{14\ MPa} \qquad \textbf{Ans.}$$

Minimum intensity of stress in the section

We also know that minimum intensity of stress in the section,

$$\sigma_{min} = \frac{P}{A}\left(1 - \frac{6e}{b}\right) = \frac{1800 \times 10^3}{18\,000}\left(1 - \frac{6 \times 10}{150}\right) \text{ N/mm}^2$$

$$= 10\,(1 - 0.4) = 6 \text{ N/mm}^2 = \mathbf{6\ MPa} \qquad \textbf{Ans.}$$



Example – 2: A rectangular column 200 mm wide and 150 mm thick is carrying a vertical load of 120 kN at an eccentricity of 50 mm in a plane bisecting the thickness. Determine the maximum and minimum intensities of stress in the section.

Solution. Given: Width (b) = 200 mm; Thickness (d) = 150 mm ; Load (P) = 120 kN = 120×10^3 N and eccentricity (e) = 50 mm.

Maximum intensity of stress in the section

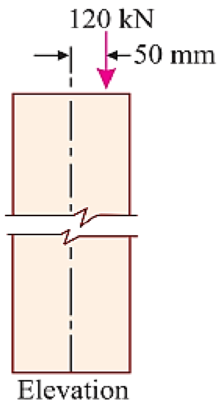
We know that area of the column,

$$A = b \times d = 200 \times 150 = 30\,000 \text{ mm}^2$$

and maximum intensity of stress in the section,

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{6e}{b}\right) = \frac{120 \times 10^3}{30\,000}\left(1 + \frac{6 \times 50}{200}\right) \text{ N/mm}^2$$

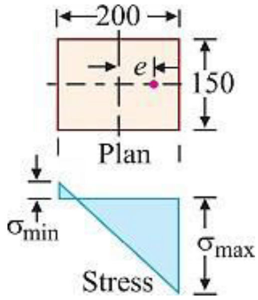
$$= 4\,(1 + 1.5) = 10 \text{ N/mm}^2 = \mathbf{10\ MPa} \qquad \textbf{Ans.}$$



Minimum intensity of stress in the section

We also know that minimum intensity of stress in the section,

$$\sigma_{min} = \frac{P}{A}\left(1-\frac{6e}{b}\right) = \frac{120\times10^3}{30\,000}\left(1-\frac{6\times50}{200}\right) \text{ N/mm}^2$$
$$= 4\,(1-1.5) = 4\,(-0.5) = -2 \text{ N/mm}^2$$
$$= 2 \text{ N/mm}^2 \text{ (tension)} = \mathbf{2\ MPa\ (tension)} \quad \textbf{Ans.}$$



Example – 3: A hollow rectangular masonry pier is 1.2 m × 0.8 m wide and 150 mm thick. A vertical load of 2 MN is transmitted in the vertical plane bisecting 1.2 m side and at an eccentricity of 100 mm from the geometric axis of the section. Calculate the maximum and minimum stress intensities in the section.

SOLUTION. Given: Outer width (B) = 1.2 m = 1.2×10^3 mm ; Load (P) = 2 MN = 2×10^6 N ; Outer thickness (D) = 0.8 m = 0.8×10^3 mm ; Thickness (t) = 150 mm and eccentricity (e) = 100 mm.

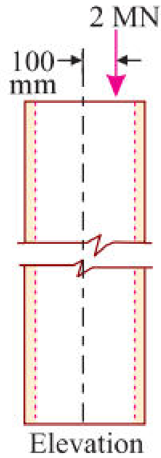
Maximum stress intensity in the section

We know that area of the pier,

$$A = (BD - bd)$$
$$= [(1.2\times10^3)\times(0.8\times10^3)] - [(0.9\times10^3)\times(0.5\times10^3)]$$
$$= (0.96\times10^6) - (0.45\times10^6) = 0.51\times10^6 \text{ mm}^2$$

and its section modulus,

$$Z = \frac{1}{6}[BD^2 - bd^2] = \frac{1}{6}[(1.2\times10^3)\times(0.8\times10^3)^2]$$
$$- [(0.9\times10^3)\times(0.5\times10^3)^2] \text{ mm}^3$$
$$= \frac{1}{6}[(768\times10^6) - (225\times10^6) = 90.5\times10^6 \text{ mm}^3$$

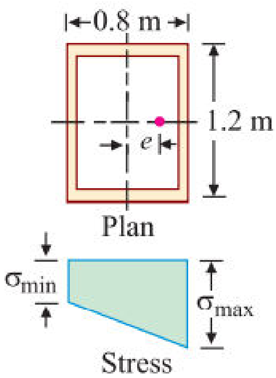


We know that moment due to eccentricity of load,

$$M = P\cdot e = (2\times10^6)\times100 = 200\times10^6 \text{ N-mm}$$

∴ Maximum stress intensity in the section,

$$\sigma_{max} = \frac{P}{A}+\frac{M}{Z} = \frac{2\times10^6}{0.51\times10^6}+\frac{200\times10^6}{90.5\times10^6} \text{ N/mm}^2$$
$$= 3.92 + 2.21 = 6.13 \text{ N/mm}^2 = \mathbf{6.13\ MPa} \quad \textbf{Ans.}$$



Minimum stress intensity in the section

We also know that minimum stress intensity in the section,

$$\sigma_{min} = \frac{P}{A}-\frac{M}{Z} = \frac{2\times10^6}{0.51\times10^6}-\frac{200\times10^6}{90.5\times10^6} \text{ N/mm}^2$$
$$= 3.92 - 2.21 = 1.71 \text{ N/mm}^2 = \mathbf{1.71\ MPa} \quad \textbf{Ans.}$$

Buckling load computation using Euler's formula (no derivation) in Columns with various end conditions:

Buckling load:

- The maximum limiting load at which the column tends to have lateral displacement or tends to buckle is called buckling or crippling load. The buckling takes place about the axis having least moment of inertia.

<u>Euler's formula</u>:

- Euler's formula is used for calculating the critical load or buckling load or crippling load for a column or strut and is as follows.

$$P_{Euler} = \frac{\pi^2 EI}{L_e{}^2}$$

Where, P = Critical load
E = Young's modulus
I = Least M.I. of section of the column
L_e = Equivalent length of the column.

<u>Equivalent length of a column (L_e)</u>:

- The distance between adjacent points of inflection is called equivalent length or effective length or simple column length.

<u>Types of End Conditions of Columns</u>:

In actual practice, there are a number of end conditions, for columns. But we shall study the Euler's column theory on the following four types of end conditions.

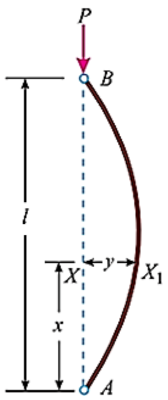
- Both ends hinged,
- Both ends fixed,
- One end is fixed and the other hinged, and
- One end is fixed and the other free

<u>Columns with Both Ends Hinged or Pinned</u>:

$$Equivalent\ length = Actual\ length$$

$$\Rightarrow\ L_e = l$$

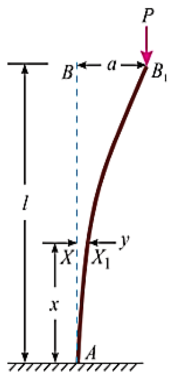
$$P = \frac{\pi^2 EI}{L_e{}^2} = \frac{\pi^2 EI}{l^2}$$



<u>Columns with One End Fixed and the Other Free</u>:

$$Here, L_e = 2l$$

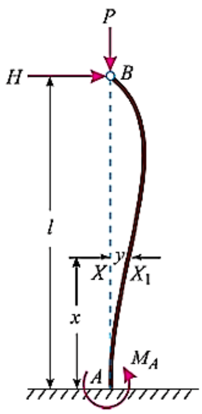
$$P = \frac{\pi^2 EI}{L_e{}^2} = \frac{\pi^2 EI}{4l^2}$$



<u>Columns with One End Fixed and the Other Hinged</u>:

$$Here, L_e = \frac{l}{\sqrt{2}}$$

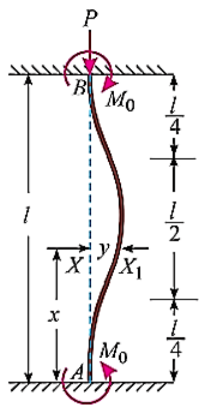
$$P = \frac{\pi^2 EI}{L_e{}^2} = \frac{2\pi^2 EI}{l^2}$$



<u>Columns with Both Ends Fixed</u>:

$$Here, L_e = \frac{l}{2}$$

$$P = \frac{\pi^2 EI}{L_e{}^2} = \frac{4\pi^2 EI}{l^2}$$



<u>Possible Short Type Question with Answers</u>:

01. What do you mean by column? (W – 2020)

- A column is a long vertical slender bar or vertical member, subjected to an axial compressive load and fixed rigidly at both ends.

02. Define eccentric load.

- A load, whose line of action does not coincide with the axis of a column or a strut, is known as an eccentric load.

03. Define buckling load.

- The maximum limiting load at which the column tends to have lateral displacement or tends to buckle is called buckling or crippling load. The buckling takes place about the axis having least moment of inertia.

04. Define equivalent length of a column.

- The distance between adjacent points of inflection is called equivalent length or effective length or simple column length.

<u>Possible Long Type Questions</u>:

01. What is meant by eccentric loading? Explain its effect on a short column. (W – 2019)
Hints: Refer article 6.1 (page no 01)

02. Define buckling load. State formula for buckling load in column with various end condition. (W – 2020)
Hints: Refer article 6.4 (page no 04 & 05)

CHAPTER 07

TORSION

LEARNING OBJECTIVES:

Assumption of pure torsion
The torsion equation for solid and hollow circular shaft
Comparison between solid and hollow shaft subjected to pure torsion

Shaft:

- Shafts are usually cylindrical in section, solid or hollow. They are made of mild steel, alloy steel and copper alloys.
- It is a structural member used to transmit mechanical power from one place to other. Shaft is subjected to torsion.
- Shafts may be subjected to the following loads:
    1. Torsional load
    2. Bending load
    3. Axial load
    4. Combination of above three loads.

Torsion:

- A shaft or circular section is said to be in pure torsion when it is subjected to equal and opposite end couples whose axes coincide with the axis of the shaft.

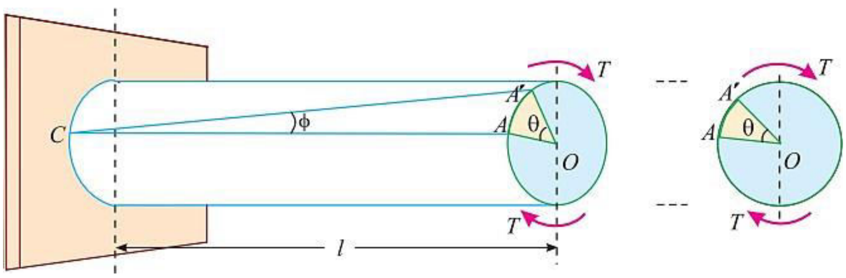
Assumption of pure torsion:

- The torsion equation is based on the following assumptions:
- The material of the shaft is homogeneous, isotropic and perfectly elastic.
- The material obeys Hooke's law and the stress remains within limit of proportionality.
- The shaft circular in section remains circular after loading.
- A plane section of shaft normal to its axis before loading remains plane after the torques have been applied.
- The twist along the length of shaft is uniform throughout.
- The distance between any two normal cross section remains the same after the application of torque.
- Maximum shear stress induced in the shaft due to application of torque does not exceed its elastic limit value.

The torsion equation for solid and hollow circular shaft:

Torsion equation for solid circular shaft:

Let, T = Maximum twisting torque
D = Diameter of the shaft
I_P = Polar moment of inertia
τ = Shear stress
G = Modulus of rigidity
θ = The angle of twist (radians)
l = Length of the shaft



- Consider a solid circular shaft fixed at one end and torque being applied at the other end. If a line CA is drawn on the shaft, it will be distorted to CA' on the application of the torque; thus, cross section will be twisted through angle θ and surface by angle ϕ.

- *Here, shear strain,*

$$\emptyset = \frac{AA'}{l}$$

- *Also,*

$$\emptyset = \frac{r}{G}$$

$$\therefore \frac{AA'}{l} = \frac{r}{G}$$

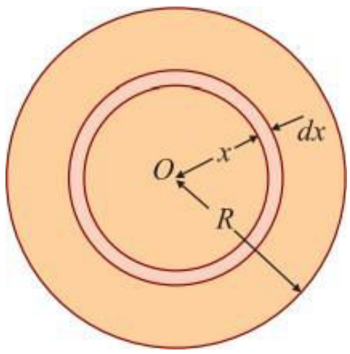
$$\Rightarrow \frac{R\theta}{l} = \frac{r}{G}$$

$$\because AA' = R \times \theta$$
$$[R \text{ being radius of the shaft}]$$

$$\therefore \frac{r}{R} = \frac{G\theta}{l} \ldots\ldots\ldots\ldots (i)$$

- Consider an elementary ring of thickness *dx* at a radius *x* and let the shear stress at this radius be τ_x.

$$i.e.\ r_x = \frac{G\theta}{l} \,.\, x$$



- The turning force on the elementary ring

$$= r_x \,.\, 2\pi x \,.\, dx$$

- Turning moment due to this turning force,

$$dT = r_x \,.\, 2\pi x \,.\, dx \times x$$

- To get total turning moment integrating both sides, we get

$$\int dT = \int_0^R r_x \,.\, 2\pi x \,.\, dx \times x$$

$$\Rightarrow \int dT = 2\pi \int_0^R \frac{G\theta}{l} \,.\, x \,.\, x^2 . dx = 2\pi \,.\, \frac{G\theta}{l} \int_0^R x^3 . dx$$

$$\Rightarrow T = 2\pi \,.\, \frac{G\theta}{l} \,.\, \left[\frac{x^4}{4}\right]_0^R = 2\pi \,.\, \frac{r}{R} \cdot \frac{R^4}{4}$$

$$\Rightarrow \boldsymbol{T} = r \,.\, \frac{\boldsymbol{\pi R^3}}{\boldsymbol{2}} = r \,.\, \frac{\boldsymbol{\pi}}{\boldsymbol{16}} \,.\, \boldsymbol{D^3}$$

$$\ldots\ldots\ldots (Strength\ of\ solid\ shaft)$$

$$Or, \quad T = \frac{r}{R} \cdot \frac{\pi R^4}{2} = \frac{r}{R} \,.\, I_P$$

$$[\because\ I_P = \frac{\pi}{32} \,.\, D^4 = \frac{\pi}{2} . R^4]$$

$$\therefore \frac{T}{I_P} = \frac{r}{R} \ldots\ldots\ldots\ldots (ii)$$

- From equation (i) and (ii), we have

$$\frac{\boldsymbol{T}}{\boldsymbol{I_P}} = \frac{r}{\boldsymbol{R}} = \frac{\boldsymbol{G\theta}}{\boldsymbol{l}}$$

- This is called **torsion equation**.

- **Note:** From the relation,

$$\frac{T}{I_P} = \frac{r}{R}$$

$$\Rightarrow T = r \times \frac{I_P}{R}$$

For a given shaft I_P and R are constants and I_P/R is thus a constant and is known as polar modulus of the shaft section.

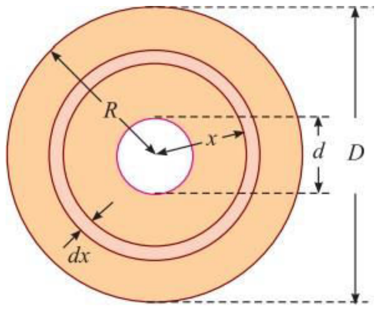
$$Thus, \qquad \boldsymbol{T} = r \times \boldsymbol{Z_P}$$

<u>Torsion equation for hollow circular shaft</u>:

- Torsion equation equally holds good for hollow circular shafts and can be established in the same way.
- Consider a hollow circular shaft subjected to a torque T
- Let, R = Outer radius of the shaft
  r = Inner radius of the shaft

- Consider an elementary ring of thickness dx at a radius x and let the shear stress at this radius be τ_x.

$$i.e.\ r_x = \frac{G\theta}{l}\,.x$$



- The turning force on the elementary ring

$$= r_x\,.\,2\pi x\,.\,dx$$

- Turning moment due to this turning force,

$$dT = r_x\,.\,2\pi x\,.\,dx \times x$$

- To get total turning moment integrating both sides, we get

$$\int dT = \int_r^R r_x\,.\,2\pi x\,.\,dx \times x$$

$$\Rightarrow \int dT = 2\pi \int_r^R \frac{G\theta}{l}\,.\,x\,.\,x^2.\,dx = 2\pi\,.\,\frac{G\theta}{l}\int_r^R x^3.\,dx$$

$$\Rightarrow T = 2\pi\,.\,\frac{G\theta}{l}\,.\,\left[\frac{x^4}{4}\right]_r^R = 2\pi\,.\,\frac{r}{R}\,.\left(\frac{R^4 - r^4}{4}\right)$$

$$\Rightarrow \boldsymbol{T} = \frac{r}{\boldsymbol{R}}\cdot\frac{\boldsymbol{\pi}}{\boldsymbol{2}}(\boldsymbol{R^4} - \boldsymbol{r^4}) = \frac{\boldsymbol{\pi}}{\boldsymbol{16}}\,.\,r\,\left(\frac{\boldsymbol{D^4} - \boldsymbol{d^4}}{\boldsymbol{D}}\right)$$

$$\ldots\ldots\ldots(Strength\ of\ hollow\ shaft)$$

$$Or, \qquad T = \frac{r}{R}\,.\,I_P$$

$$[\because\ I_P = \frac{\pi}{32}\,(D^4 - d^4) = \frac{\pi}{2}\,(R^4 - r^4)]$$

$$\therefore \quad \frac{T}{I_P} = \frac{r}{R} \ldots\ldots\ldots\ldots (iii)$$

- From equation (*i*) and (*iii*), we have

$$\frac{\boldsymbol{T}}{\boldsymbol{I_P}} = \frac{r}{\boldsymbol{R}} = \frac{\boldsymbol{G\theta}}{\boldsymbol{l}}$$

Torsional rigidity:

- From the relation,

$$\frac{T}{I_P} = \frac{G\theta}{l}$$

- We have,

$$\theta = \frac{Tl}{GI_P}$$

- Since G, l and I_P are constant for a given shaft, θ the angle of twist is directly proportional to the twisting moment. The quantity $\frac{\boldsymbol{GIP}}{\boldsymbol{l}}$ is known as torsional rigidity and is represented by k or μ.
- From the above relation, we have

$$\boldsymbol{k} = \frac{\boldsymbol{GI_P}}{\boldsymbol{l}} = \frac{\boldsymbol{T}}{\boldsymbol{\theta}}$$

Power transmitted by the shaft:

- Consider a force *F* newton acting tangentially on the shaft of radius *R*. If the shaft due to this turning moment ($F \times R$) starts rotating at *N* r.p.m. then,

$$Work\ supplied\ to\ the\ shaft/sec = Force \times distance\ moved/sec$$
$$= F \times \frac{2\pi RN}{60}\ Nm/s$$
$$\boldsymbol{P} = \frac{\boldsymbol{F \times 2\pi RN}}{\boldsymbol{60}} = \frac{\boldsymbol{2\pi NT}}{\boldsymbol{60}}\boldsymbol{watt}$$

$\ldots\ldots\ldots\ldots (\because T = F \times R)$

Where, *T* = Mean/average torque

Example – 1: A circular shaft of 60 mm diameter is running at 150 r.p.m. If the shear stress is not to exceed 50 MPa, find the power which can be transmitted by the shaft.

Solution. Given: Diameter of the shaft (D) = 60 mm; Speed of the shaft (N) = 150 r.p.m. and maximum shear stress (τ) = 50 MPa = 50 N/mm^2.

- We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times r \times D^3$$
$$= \frac{\pi}{16} \times 50 \times 60^3$$
$$= 2.12 \times 10^6\ N.mm = 2.12\ kN.m$$

- and power which can be transmitted by the shaft,

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 150 \times 2.12}{60} = \boldsymbol{33.3\ kW}$$

Example - 2: A hollow shaft of external and internal diameters as 100 mm and 40 mm is transmitting power at 120 r.p.m. Find the power the shaft can transmit, if the shearing stress is not to exceed 50 MPa.
Solution. Given: External diameter (D) = 100 mm; Internal diameter (d) = 40 mm; Speed of the shaft (N) = 120 r.p.m. and allowable shear stress (τ) = 50 MPa = 50 N/mm^2.

- We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times r \times (\frac{D^4 - d^4}{D})$$

$$= \frac{\pi}{16} \times 50 \times (\frac{100^4 - 40^4}{100})$$

$$= 9.56 \times 10^6 \; N.mm = 9.56 \; kN.m$$

- and power which can be transmitted by the shaft,

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 120 \times 9.56}{60} = \mathbf{120 \; kW}$$

Example – 3: A solid circular shaft of 100 mm diameter is transmitting 120 kW at 150 r.p.m. Find the intensity of shear stress in the shaft.
Solution. Given: Diameter of the shaft (D) = 100 mm; Power transmitted (P) = 120 kW and speed of the shaft (N) = 150 r.p.m.

- We know that power transmitted by the shaft (P),

$$120 = \frac{2\pi NT}{60} = \frac{2\pi \times 150 \times T}{60}$$

$$\Rightarrow T = \frac{120 \times 60}{2\pi \times 150} = 7.64 \; kN.m$$

$$= 7.64 \times 10^6 \; N.mm$$

- We also know that torque transmitted by the shaft (T),

$$7.64 \times 10^6 = \frac{\pi}{16} \times r \times D^3 = \frac{\pi}{16} \times r \times 100^3$$

$$\Rightarrow r = \frac{7.64 \times 16}{\pi} = 39 \; N/mm^2 = \mathbf{39 \; MPa}$$

Example – 4: A hollow shaft is to transmit 200 kW at 80 r.p.m. If the shear stress is not to exceed 60 MPa and internal diameter is 0.6 of the external diameter, find the diameters of the shaft.
Solution. Given: Power (P) = 200 kW; Speed of shaft (N) = 80 r.p.m.; Maximum shear stress (τ) = 60 MPa = 60 N/mm^2 and internal diameter of the shaft (d) = 0.6D (where D is the external diameter in mm).

- We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times r \times (\frac{D^4 - d^4}{D}) = \frac{\pi}{16} \times 60 \times (\frac{D^4 - (0.6D)^4}{D})$$

$$= 10.3 \; D^3 \; N.mm = 10.3 \times 10^{-6} \; D^3 \; kN.m$$

- We also know that power transmitted by the shaft (P),

$$200 = \frac{2\pi NT}{60} = \frac{2\pi \times 80 \times (10.3 \times 10^{-6} \; D^3)}{60} = 86.3 \times 10^{-6} \; D^3$$

$$\Rightarrow \; D^3 = \frac{200}{86.3 \times 10^{-6}} = 2.32 \times 10^6 \; mm^3$$

$$\Rightarrow D = 1.32 \times 10^2 = \mathbf{132 \; mm}$$

$$And, \qquad d = 0.6 \; D = 0.6 \times 132 = \mathbf{79.2 \; mm}$$

Comparison between solid and hollow shaft subjected to pure torsion:

(A) Comparison by strength:

- In this case it is assumed that both the shafts have same length, material, same weight and hence the same maximum shear stress.

- We know that,

$$T_S = r \times \frac{\pi}{16} \times (D_S)^3$$

$$T_H = r \times \frac{\pi}{16} \times \left[\frac{(D_H)^4 - (d_H)^4}{D_H}\right]$$

$$\therefore \frac{Strength\ of\ hollow\ shaft}{Strength\ of\ solid\ shaft} = \frac{T_H}{T_S} = \frac{r \times \frac{\pi}{16} \times \left[\frac{(D_H)^4 - (d_H)^4}{D_H}\right]}{r \times \frac{\pi}{16} \times (D_S)^3}$$

$$\Rightarrow \frac{T_H}{T_S} = \frac{(D_H)^4 - (d_H)^4}{D_H\ .\ (D_S)^3} \ldots\ldots\ldots (i)$$

$$Let, \quad \frac{D_H}{d_H} = n$$

$$\Rightarrow D_H = nd_H.$$

- Substituting $D_H = nd_H$ in equation (i), we get

$$\frac{T_H}{T_S} = \frac{n^4 d_H{}^4 - d_H{}^4}{nd_H D_S{}^3} = \frac{d_H{}^4(n^4 - 1)}{nd_H D_S{}^3} = \frac{d_H{}^3(n^4 - 1)}{nD_S{}^3} \ldots\ldots\ldots (ii)$$

- As the weight, material and length of both the shafts are same,

  Cross sectional area of solid shaft = Cross sectional area of hollow shaft

$$\frac{\pi}{4} D_S{}^2 = \frac{\pi}{4}(D_H{}^2 - d_H{}^2) \quad or \quad D_S = \sqrt{D_H{}^2 - d_H{}^2}$$

$$\Rightarrow D_S{}^3 = (D_H{}^2 - d_H{}^2)\sqrt{D_H{}^2 - d_H{}^2}$$

$$\Rightarrow D_S{}^3 = (n^2 d_H{}^2 - d_H{}^2)\sqrt{n^2 d_H{}^2 - d_H{}^2}$$

$$\Rightarrow D_S{}^3 = d_H{}^3(n^2 - 1)\sqrt{n^2 - 1}$$

- Substituting the value of $D_S{}^3$ in equation (ii), we get

$$\frac{T_H}{T_S} = \frac{d_H{}^3(n^4 - 1)}{n\, d_H{}^3(n^2 - 1)\sqrt{n^2 - 1}}$$

$$= \frac{(n^2 + 1)(n^2 - 1)}{n\,(n^2 - 1)\sqrt{n^2 - 1}} = \frac{n^2 + 1}{n\sqrt{n^2 - 1}}$$

- Since $D_H > d_H$ and $\frac{D_H}{d_H} = n$, it is thus obvious that the value of 'n' is greater than unity.
- Suppose, $n = 2$

  Then,

$$\frac{T_H}{T_S} = \frac{n^2 + 1}{n\sqrt{n^2 - 1}} = \frac{2^2 + 1}{2\sqrt{2^2 - 1}} = 1.44$$

- This shows that the torque transmitted by the hollow shaft is greater than the solid shaft, thereby proving that the hollow shaft is stronger than the solid shaft.

(B) <u>Comparison by weight</u>:

- In this case it is assumed that both the shafts have same length and material. Now, if the torque applied to both shafts is same, then, the maximum shear stress will also be same in both the cases.
- Now,

$$\frac{Weight\ of\ hollow\ shaft}{Weight\ of\ solid\ shaft} = \frac{W_H}{W_S} = \frac{A_H}{A_S}$$

$$= \frac{\frac{\pi}{4}(D_H{}^2 - d_H{}^2)}{\frac{\pi}{4}D_S{}^2} = \frac{D_H{}^2 - d_H{}^2}{D_S{}^2} \ldots\ldots\ldots\ldots (i)$$

$$Let,\quad \frac{D_H}{d_H} = n$$

$$\Rightarrow D_H = nd_H.$$

- Substituting $D_H = nd_H$ in equation (i), we get

$$\frac{W_H}{W_S} = \frac{n^2 d_H{}^2 - d_H{}^2}{D_S{}^2} = \frac{d_H{}^2(n^2 - 1)}{D_S{}^2} \ldots\ldots\ldots\ldots (ii)$$

- Torque applied in both the cases is same i.e., $T_S = T_H$

$$r \times \frac{\pi}{16} \times (D_S)^3 = r \times \frac{\pi}{16} \times [\frac{(D_H)^4 - (d_H)^4}{D_H}]$$

$$\Rightarrow D_S{}^3 = \frac{(D_H)^4 - (d_H)^4}{D_H} = \frac{n^4 d_H{}^4 - d_H{}^4}{nd_H} = \frac{d_H{}^3(n^4 - 1)}{n}$$

$$\Rightarrow D_S = d_H [\frac{n^4 - 1}{n}]^{1/3} \quad \Rightarrow D_S{}^2 = d_H{}^2 [\frac{n^4 - 1}{n}]^{2/3}$$

- Substituting the value of $D_S{}^2$ in equation (ii), we have

$$\frac{W_H}{W_S} = \frac{d_H{}^2(n^2 - 1)}{d_H{}^2 [\frac{n^4 - 1}{n}]^{2/3}} = \frac{(n^2 - 1)n^{2/3}}{(n^4 - 1)^{2/3}}$$

- If $n = 2$ then,

$$\frac{W_H}{W_S} = \frac{(2^2 - 1) \times (2)^{2/3}}{(2^4 - 1)^{2/3}} = 0.7829$$

- It shows that for same material, length and given torque, weight of hollow shaft will be less. So, hollow shafts are economical compared to solid shafts as regards torsion.

Possible Short Type Question with Answers:

01. Define torsion. (W – 2020)

- A shaft or circular section is said to be in pure torsion when it is subjected to equal and opposite end couples whose axes coincide with the axis of the shaft.

02. What is polar moment of inertia? (W – 2019)

- The moment of inertia of a plane area, with respect to an axis perpendicular to the plane of the figure, is called polar moment of inertia with respect to the point, where the axis intersects the plane.

Possible Long Type Questions:

01. A solid circular shaft of 80 mm diameter is required to transmit power at 120 r.p.m. If the shear stress is not to exceed 40 MPa, find the power transmitted by the shaft. **(W – 2019)**
Hints: Refer Example – 01 (page no 04)

02. Find the maximum shear stress induced in a solid circular shaft of diameter 15 cm when the shaft transmits 150 kW power at 180 r.p.m. **(W – 2020)**
Hints: Refer Example – 03 (page no 05)

03. What are the assumptions of pure torsion?
Hints: Refer Article – 7.1 (page no 01)

04. Derive $\frac{T}{I_P} = \frac{\tau}{R} = \frac{G\theta}{l}$ for pure torsion.

Where, T = Maximum twisting torque
R = Radius of the shaft
I_P = Polar moment of inertia
τ = Shear stress
G = Modulus of rigidity
θ = The angle of twist (radians)
l = Length of the shaft

Hints: Refer Article – 7.2