LECTURES NOTE

SUB: ENGINEERING MATHEMATICS I

NAME OF FACULTY: MANAS KUMAR MAHALIK

(Lecturer in Mathematics)

GOVERNMENT POLYTECHNIC, BHADRAK

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HOD ,Math&Sc

14-8-23
Academic Co-ordinator

Principal
Govt. Polytechnic, Bhadrak

INTRODUCTION TO MATRIX :

Equation :-

It is a statement of two Mathematical expression indicated by an equal (=) sign.

for example → $3x^2 - 5 = 4x + 3$

where $3x^2 - 5$ and $4x+3$ are two Mathematical expressions.

Types :-

There are different types of equations, such as

1. Linear Equation (Degree 1)
2. Quadratic Equation (Degree 2)
3. Cubic Equation (Degree 3)
4. Quartic Equation (Degree 4)
5. Quintic Equation (Degree 5)

and so... on...

LINEAR EQUATIONS

Linear Equations are the equations of degree '1'.

Such as.

(i) $4x+3 = 5x+9$ (Linear equation of one Variable)

(ii) $3x+4y=9$ (Linear equation of two Variables)

(iii) $2x-3y+4z=0$ (Linear equation of three Variables)

Solution of linear Equations

(1) Solution of linear eq^n of one Variable.

Q:-1 Solve $9x+4 = 11x-5$

Sol^n:- Given $9x+4 = 11x-5$

$$\Rightarrow 9x-11x = -5-4$$

$$\Rightarrow -2x = -9$$

$$\Rightarrow x = \frac{-9}{-2} = \frac{9}{2}$$

(2) Solution of linear eq^{n}s of two Variables

Q:-1 Solve $x+y=3$

$x+3y=5$

Now this type of linear system of equations can be solved by different Methods, like

1. Substitution Method.
2. Elimination Method.
3. cross Multiplication Method.
4. Graphical Method.

1. Substitution Method:

Sol^n:- Given $x+y=3$ ——(i)

$x+3y=5$ ——(ii)

from eq^n (1) we get $x=3-y$

putting $x=3-y$ in eq^n (ii)

eq^n (ii) becomes

$$\Rightarrow \cancel{x} \; 3-y+3y=5$$

$$\Rightarrow 3+2y=5$$

$$\Rightarrow 2y=2$$

$$\Rightarrow \boxed{y=1}$$

putting $y=1$ in eqⁿ ①

$\Rightarrow x+1=3$

$\Rightarrow \boxed{x=2}$

(ii) Elimination Method :-

Solⁿ :- Given $x+y=3$ —— (i)

$x+3y=5$ —— (ii)

As already the coefficients of variable 'x' are equal so, Substract eqⁿ (ii) from eqⁿ (i)

i.e.

$$\begin{array}{r} \cancel{x}+y=3 \\ \cancel{+x}+3y=5 \\ (-)\quad(-)\quad(-) \\ \hline -2y=-2 \end{array}$$

$\Rightarrow \boxed{y=1}$

putting the value of 'y' in eqⁿ ①

we get $\boxed{x=2}$

(iii) cross Multiplication Method :-

Solⁿ :- Given $x+y=3$

$x+3y=5$

The above equations can be rewritten as

$x+y-3=0$ $[a_1=1,\ b_1=1,\ c_1=-3]$

$x+3y-5=0$ $[a_2=1,\ b_2=3,\ c_2=-5]$

By Method of cross Multiplication,

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

Substituting the values in the above equation.

$$\Rightarrow \frac{x}{1(-5)-3(-3)}=\frac{y}{(-3)(1)-(-5)(1)}=\frac{1}{1(3)-1(1)}$$

$$\Rightarrow \frac{x}{-5+9}=\frac{y}{-3+5}=\frac{1}{3-1}$$

$$\Rightarrow \frac{x}{4}=\frac{y}{2}=\frac{1}{2}$$

consider $\frac{x}{4}=\frac{1}{2} \Rightarrow x=\frac{4}{2} \Rightarrow \boxed{x=2}$

Again consider $\frac{y}{2}=\frac{1}{2} \Rightarrow y=\frac{2}{2} \Rightarrow \boxed{y=1}$

Graphical Method :-

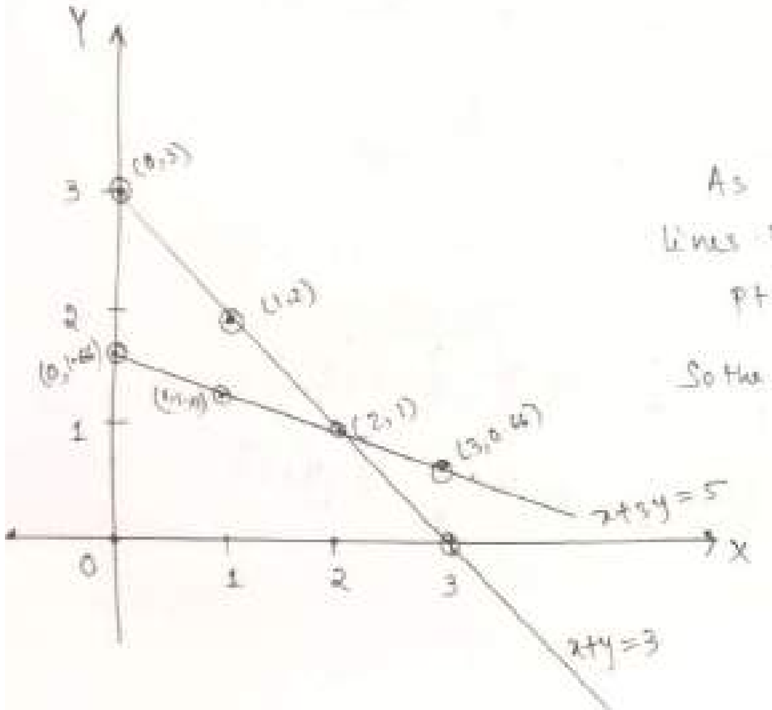
Given eq^n's are $x+y=3$
$x+3y=5$

Now for $x+y=3$

x	0	1	2	3
y	3	2	1	0

for $x+3y=5$

x	0	1	2	3
y	1.66	1.33	1	0.66



As both the lines intersect at the pt (2,1)

So the sol^n is $\boxed{x=2}$ $\boxed{y=1}$

* There is a another Method for solving linear System of equations known as Matrix Method

Application of Matrices :-

Basically Matrices is an essential Mathematical tool which helps

(i) To represent large System of linear equations

(ii) To find the solution of such equations

MATRIX :-

Definition :-

Matrix means arrangement of numbers in some rows and columns in a rectangular shape.
(horizontal lines) (Vertical lines)

for example :-

$$A = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 4 & 9 & 6 & 5 \\ -5 & 3 & 0 & 2 \\ 7 & 9 & -6 & 8 \end{bmatrix} \begin{array}{c} \to R_1 \\ \to R_2 \\ \to R_3 \end{array} \end{array}$$

* Matrices are always denoted by capital letters

* Numbers in the Matrix, known as elements of Matrix.

Order :-

The number of rows and columns that a Matrix has is called its order or dimension.

Basically rows are listed first and columns second. And is written in the form.

$m \times n$, means m rows and n columns.

for example :-

$$A = \begin{bmatrix} 3 & 4 & 9 \\ 0 & -6 & 5 \end{bmatrix}$$

Here the above Matrix has 2 rows and 3 colum

So the order of $A = 2 \times 3$

General form of a Matrix of order $m \times n$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

Here the elements are written in the form

a_{ij} where i - rows
j = columns.

So a_{32} means element in the 3^{rd} row and 2^{nd} column.

for example :-

Q.1 write down a Matrix of order 4 × 3.

Sol^n

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}_{4\times 3}$$

Q.2 write down a Matrix of order 2×3 s.t. $a_{ij} = i+j$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3}$$

As $a_{ij} = i+j$

So, $a_{11} = 1+1 = 2$ $\quad a_{21} = 2+1 = 3$

$a_{12} = 1+2 = 3$ $\quad a_{22} = 2+2 = 4$

$a_{13} = 1+3 = 4$ $\quad a_{23} = 2+3 = 5$

So $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}_{2\times 3}$

TYPES OF MATRIX :

① Row Matrix :-

A Matrix of order $1 \times n$ (Matrix having one row) is call[ed] a row Matrix.

for example

$A = \begin{bmatrix} 1 & 4 & 0 & 9 \end{bmatrix}_{1\times 4}$

$B = \begin{bmatrix} 4 & -5 & 6 \end{bmatrix}_{1\times 3}$

$C = \begin{bmatrix} -7 \end{bmatrix}_{1\times 1}$

② column Matrix :-

A Matrix of order $m \times 1$ (Matrix having one column) is called column Matrix.

for example

$A = \begin{bmatrix} 4 \\ 9 \\ -5 \end{bmatrix}_{3\times 1}$

$B = \begin{bmatrix} 4 \\ 0 \\ 9 \\ 1 \\ 6 \end{bmatrix}_{5\times 1}$ $\quad C = \begin{bmatrix} 9 \end{bmatrix}_{1\times 1}$

(3) Square Matrix :-

If the number of rows is equal to the number of columns, then the Matrix is called square Matrix.

for example :-

$$A = \begin{bmatrix} 4 & 0 & -5 \\ 2 & 3 & 4 \\ 7 & 2 & 1 \end{bmatrix}_{3\times 3} \quad \text{(order 3)}$$

$$B = \begin{bmatrix} 4 & 2 \\ 9 & 8 \end{bmatrix}_{2\times 2} \quad \text{(order 2)}$$

$$A = \begin{bmatrix} 5 \end{bmatrix}_{1\times 1} \quad \text{(order 1)}$$

(4) Digonal Matrix :-

A square Matrix is said to be a diagonal Matrix if its non-diagonal elements are zero.

for example :-

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$$

(5) Scalar Matrix :-

A diagonal Matrix is said to be scalar Matrix if all the diagonal elements are equal.

for example:

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times 3}$$

(6) Identity / Unit Matrix :-

A square Matrix in which all the diagonal elements are '1' and rest of the elements are zero.

for example :-

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$I_1 = \begin{bmatrix} 1 \end{bmatrix}_{1\times 1}$$

(7) Zero/Null Matrix :-

A Matrix is said to be Null Matrix if all the elements are zero.

for example :

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$$

(8) Upper-Triangular Matrix :-

A square Matrix is called a upper triangular Matrix if all the elements below the main diagonal are zero.

for example :-

$$A = \begin{bmatrix} 4 & 9 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & 8 \end{bmatrix}_{3\times 3}$$

(9) Lower-Triangular Matrix :-

A square Matrix is called a lower triangular Matrix if all the elements above the main diagonal are zero.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 9 & 6 & 0 \\ 5 & 2 & 1 \end{bmatrix}$$

Equality of two Matrices

Two Matrices 'A' & 'B' are said to be equal iff (i) order of A = order of B

(ii) Each element of A is equal to the corresponding element of B

for example :-

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Here (i) order of A = order of B = 2x2

and if $a = x$, $b = y$, $c = z$ & $d = w$

then $A = B$.

Transpose of a Matrix

Let 'A' be a Matrix, then its transpose is obtained by converting rows into columns or columns into rows and denoted as A^T.

for examples :- $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & -2 & 1 \end{bmatrix}_{2\times 3}$

Then $A^T = \begin{bmatrix} 2 & 3 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}_{3\times 2}$

Operations on Matrix :-

(i) Addition of two Matrix :-

We can add two Matrices if their order is same

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

Then $A+B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$= \begin{bmatrix} a+x & b+y \\ c+z & d+w \end{bmatrix}$

Note :- when we add two matrix A and B, we add each element of A with the corresponding element of B.

(ii) Substraction :-

Similarly we can substract two matrices if their order is same.

for example :- $A-B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$= \begin{bmatrix} a-x & b-y \\ c-z & d-w \end{bmatrix}$

(iii) Multiplication :-

Case I :- Multiplication of a Matrix by a number.

Let A be a Matrix, 'K' be a number.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $KA = \begin{bmatrix} Ka & Kb \\ Kc & Kd \end{bmatrix}$

i.e. when a number 'K' is multiplied to Matrix, actually it is multiplied with each and every element of the matrix.

Case II :- Multiplication of two Square Matrices.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}_{2\times2}$ be two Square Matrices of order 2

Then $AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ by Row-column Method.

$= \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix}$

Q.1 find AB and BA

where $A = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$

Solution :- $$AB = \overset{\xrightarrow{\text{row}}}{\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}} \underset{\downarrow}{\text{column}} \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (2\times3)+(1\times5) & (2\times2)+(1\times4) \\ (-3\times3)+(0\times5) & (-3\times2)+(0\times4) \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 8 \\ -9 & -6 \end{bmatrix}$$

Again $$BA = \overset{\xrightarrow{\text{row}}}{\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}} \underset{\downarrow}{\text{column}} \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (3\times2)+(2\times-3) & (3\times1)+(2\times0) \\ (5\times2)+(4\times-3) & (5\times1)+(4\times0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}$$

Q.2 find AB

where $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 4 \\ 1 & 1 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 & 2 \\ -3 & 1 & 1 \\ -1 & 0 & 4 \end{bmatrix}$

Solution :- $$AB = \overset{\xrightarrow{\text{row}}}{\begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 4 \\ 1 & 1 & 5 \end{bmatrix}} \underset{\downarrow}{\text{column}} \begin{bmatrix} 2 & -1 & 2 \\ -3 & 1 & 1 \\ -1 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (2\times2)+(0\times-3)+(1\times-1) & (2\times-1)+(0\times1)+(1\times0) & (2\times2)+(0\times1)+(1\times \\ (3\times2)+(-2\times-3)+(4\times-1) & (3\times-1)+(-2\times1)+(4\times0) & (3\times2)+(-2\times1)+(4\times \\ (1\times2)+(1\times-3)+(5\times-1) & (1\times-1)+(1\times1)+(5\times0) & (1\times2)+(1\times1)+(5\times \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+(-1) & (-2)+0+0 & 4+0+4 \\ 6+6+(-4) & (-3)+(-2)+0 & 6+(-2)+16 \\ 2+(-3)+(-5) & (-1)+1+0 & 2+1+20 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 8 \\ 8 & -5 & 20 \\ -6 & 0 & 23 \end{bmatrix}$$

Case III :- Multiplication of two non-square Matrices.

Q:-1 $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 4 & 2 \\ -1 & 6 \\ 0 & 3 \end{bmatrix}_{3\times 2}$

Then evaluate AB and BA.

Solution :- Given $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ -1 & 6 \\ 0 & 3 \end{bmatrix}$

$$\text{Then } AB = \overset{\xrightarrow{\text{row}}}{\begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \end{bmatrix}} \ \underset{\text{column}}{\downarrow} \begin{bmatrix} 4 & 2 \\ -1 & 6 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (2\times 4)+(3\times -1)+(-1\times 0) & (2\times 2)+(3\times 6)+(-1\times 3) \\ (4\times 4)+(0\times -1)+(2\times 0) & (4\times 2)+(0\times 6)+(2\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 8+(-3)+0 & 4+18+(-3) \\ 16+0+0 & 8+0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 19 \\ 16 & 14 \end{bmatrix}_{2\times 2}$$

$$\text{Again } BA = \begin{bmatrix} 4 & 2 \\ -1 & 6 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} (4\times 2)+(2\times 4) & (4\times 3)+(2\times 0) & (4\times -1)+(2\times 2) \\ (-1\times 2)+(6\times 4) & (-1\times 3)+(6\times 0) & (-1\times -1)+(6\times 2) \\ (0\times 2)+(3\times 4) & (0\times 3)+(3\times 0) & (0\times -1)+(3\times 2) \end{bmatrix}$$

$$= \begin{bmatrix} 8+8 & 12+0 & -4+4 \\ -2+24 & -3+0 & 1+12 \\ 0+12 & 0+0 & 0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 12 & 0 \\ 22 & -3 & 13 \\ 12 & 0 & 6 \end{bmatrix}_{3\times 3}$$

Imp NOTE :- To multiply an $m\times n$ Matrix by $p\times q$ Matrix, n must be equal to p (i.e. $\boxed{n=p}$) and resultant Matrix will be of order $\boxed{m\times q}$.

Q:-2 If $A = \begin{bmatrix} 2 & 5 & 7 \\ -1 & 6 & 3 \end{bmatrix}_{2\times3}$ & $B = \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix}_{3\times1}$

Then find AB and BA.

Solution :- $A = \begin{bmatrix} 2 & 5 & 7 \\ -1 & 6 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 5 & 7 \\ -1 & 6 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix}$$

(i) Here AB is possible as no. of column of A = no. of rows of B = 3

(ii) By multiplying a 2x3 by a 3x1 we will get a 2x1 matrix)

$$= \begin{bmatrix} (2\times4) + (5\times9) + (7\times-3) \\ (-1\times4) + (6\times9) + (3\times-3) \end{bmatrix}$$

$$= \begin{bmatrix} 8+45+(-21) \\ (-4)+54+(-9) \end{bmatrix}$$

$$= \begin{bmatrix} 32 \\ 41 \end{bmatrix}_{2\times1}$$

Again $BA = \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} \begin{bmatrix} 2 & 5 & 7 \\ -1 & 6 & 3 \end{bmatrix}$

Here BA is not possible.

as we have B is of order 3x1 &

A is of order 2x3

And number of columns of B is not equal to the number of rows of A.

(iv) Divison :-

Case I :- Divison of a Matrix by a number.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a Matrix.

and k be a number.

Then $\frac{A}{k} = \frac{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}{k} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \frac{1}{k}$

Note

Instead of dividing k we can multiply $\frac{1}{k}$ (i.e. multiplicative inverse of k) to the Matrix.

$= \begin{bmatrix} a(\frac{1}{k}) & b(\frac{1}{k}) \\ c(\frac{1}{k}) & d(\frac{1}{k}) \end{bmatrix}$

$= \begin{bmatrix} \frac{a}{k} & \frac{b}{k} \\ \frac{c}{k} & \frac{d}{k} \end{bmatrix}$

Case II :- Divison of a Matrix by another Matrix.

→ Technically there is no such thing as Matrix division.

→ Dividing a Matrix by another Matrix is undefined.

So if $[A] \div [B]$ is undefined

but we can solve it by multiplying inverse of Matrix $[B]$ with $[A]$

i.e. $[A] \times [B]^{-1}$, which is the closest equivalent of divison.

How to find Multiplicative inverse of a Matrix :-

Let A be a Matrix, then its Multiplicative inverse is denoted by A^{-1}

and is obtained as $\boxed{A^{-1} = \frac{Adj(A)}{|A|}}$

Determinant :-

Basically Determinant is a Scalar Value ~~derived~~ calculated from the elements of a square Matrix.

Case I Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a order 2 Square Matrix,

Then $|A| = ad - bc$

i.e. Multiplying the elements of main diagonal minus multiply elements of anti diagonal

Case II

let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a order 3 Square Matrix.

Then determinant of A is denoted as $|A|$.

and is defined as.

$$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - eg).$$

* To find the determinant of 3X3 Matrix first choose any row ~~or~~ or any column.
* Take the first element, cross out the row/column it belong to, find the determinant of the remaining 2X2 Matrix and multiply it with the choosen element.
* While doing this calculation always refer to the Matrix sign chart.

i.e $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

* Repeat the process for the other two elements of the choosen row/column

Q:-1 find the determinant of the Matrix

$A = \begin{bmatrix} 4 & 2 \\ -7 & 8 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 2 \\ -7 & 8 \end{vmatrix}$$

$$= 32 - (-7 \times 2)$$

$$= 32 - (-14) = 32 + 14 = 46$$

Q:-2 find the determinant of the Matrix $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 4 & 9 \\ 5 & -2 & 8 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 4 & -2 \\ 3 & 4 & 9 \\ 5 & -2 & 8 \end{vmatrix}$$

Sign chart

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Choose R_1 to find the value of the above determinant

$$= 1\begin{vmatrix} 4 & 9 \\ -2 & 8 \end{vmatrix} - 4\begin{vmatrix} 3 & 9 \\ 5 & 8 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix}$$

$$= 1(32 - (-18)) - 4(24 - 45) - 2(-6 - 20)$$

$$= 1(32 + 18) - 4(-21) - 2(-26) = 50 + 84 + 52 = 186 \text{ (Ans)}$$

Q.2 Find the determinant of the Matrix

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 5 & 0 & -6 \\ 9 & 3 & 7 \end{bmatrix}$$

Solution :-

$$|A| = \begin{vmatrix} 2 & -1 & 4 \\ 5 & 0 & -6 \\ 9 & 3 & 7 \end{vmatrix}$$

sign chart

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

choose C_1 to find the determinant of the

$$= 2\begin{vmatrix} 0 & -6 \\ 3 & 7 \end{vmatrix} - 5\begin{vmatrix} -1 & 4 \\ 3 & 7 \end{vmatrix} + 9\begin{vmatrix} -1 & 4 \\ 0 & -6 \end{vmatrix}$$

$$= 2(0-(-18)) - 5(-7-12) + 9(6-0)$$

$$= 2(18) - 5(-19) + 9(6)$$

$$= 36 + 95 + 54$$

$$= 185 \text{ (Ans).}$$

Adjoint of a Matrix :-

Adjoint of a Matrix A is the Transpose of the co-factor Matrix of A.

i.e. $\text{Adj}(A) = [\text{co-factor}(A)]^T$

Co-factor

Co-factor of an element a_{ij} is denoted by C_{ij} (or A_{ij}) and is obtained as.

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

where M_{ij} is the Minor of the element a_{ij}.

Minor

"Minor of an element is obtained by eliminating the row and column it belongs to".

Case I :- Consider a Order 2 determinant.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$$

Minor of a_{11}, $M_{11} = a_{22}$

Minor of a_{12}, $M_{12} = a_{21}$

Minor of a_{21}, $M_{21} = a_{12}$

Minor of a_{22}, $M_{22} = a_{11}$

Case II :- Consider a order 3 determinant.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Minor of a_{11}, $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

Minor of a_{12}, $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$

Minor of a_{13}, $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$

Minor of a_{21}, $M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{13}a_{32}$

Minor of a_{22}, $M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{13}a_{31}$

Minor of a_{23}, $M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{12}a_{31}$

Minor of a_{31}, $M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}a_{23} - a_{13}a_{22}$

Minor of a_{32}, $M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11}a_{23} - a_{13}a_{21}$

Minor of a_{33}, $M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Q.1 find the Minor of Matrix of $A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$

Sol^n Given $A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$

$M_{11} = 6$

$M_{12} = -1$

$M_{21} = 4$

$M_{22} = 2$

Then Matrix of Minors

$$= \begin{bmatrix} 6 & -1 \\ 4 & 2 \end{bmatrix}$$

Q.2 Find the Minors of the Matrix

$$A = \begin{bmatrix} 5 & 4 & -2 \\ 2 & 0 & 9 \\ -3 & 6 & 7 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 0 & 9 \\ 6 & 7 \end{vmatrix} = 0 - 54 = -54$$

$$M_{12} = \begin{vmatrix} 2 & 9 \\ -3 & 7 \end{vmatrix} = 14 - (-27) = 14 + 27 = 41$$

$$M_{13} = \begin{vmatrix} 2 & 0 \\ -3 & 6 \end{vmatrix} = 12 - 0 = 12$$

$$M_{21} = \begin{vmatrix} 4 & -2 \\ 6 & 7 \end{vmatrix} = 28 - (-12) = 28 + 12 = 40$$

$$M_{22} = \begin{vmatrix} 5 & -2 \\ -3 & 7 \end{vmatrix} = 35 - 6 = 29$$

$$M_{23} = \begin{vmatrix} 5 & 4 \\ -3 & 6 \end{vmatrix} = 30 - (-12) = 30 + 12 = 42$$

$$M_{31} = \begin{vmatrix} 4 & -2 \\ 0 & 9 \end{vmatrix} = 36 - 0 = 36$$

$$M_{32} = \begin{vmatrix} 5 & -2 \\ 2 & 9 \end{vmatrix} = 45 - (-4) = 45 + 4 = 49$$

$$M_{33} = \begin{vmatrix} 5 & 4 \\ 2 & 0 \end{vmatrix} = 0 - 8 = -8$$

Then Matrix of Minors is given by

$$= \begin{bmatrix} -54 & 41 & 12 \\ 40 & 29 & 42 \\ 36 & 49 & -8 \end{bmatrix}$$

Co-factor $= (-1)^{i+j} M_{ij}$

So if we have to find co-factors for the same question, then

$$C_{11} = (-1)^{1+1} \cdot M_{11} = (-1)^2 \cdot (-54) = 1(-54) = -54$$

$$C_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^3 \cdot 41 = (-1)(41) = -41$$

$$C_{13} = (-1)^{1+3} \cdot M_{13} = (-1)^4 \cdot 12 = 1(12) = 12$$

$C_{21} = (-1)^{2+1} \times M_{21} = (-1)^3 \cdot 40 = (-1)40 = -40$

$C_{22} = (-1)^{2+2} \cdot M_{22} = (-1)^4 \cdot 29 = (1)(29) = 29$

$C_{23} = (-1)^{2+3} \cdot M_{23} = (-1)^5 \cdot 42 = (-1)(42) = -42$

$C_{31} = (-1)^{3+1} \cdot M_{31} = (-1)^4 \cdot 36 = 1 \cdot (36) = 36$

$C_{32} = (-1)^{3+2} \cdot M_{32} = (-1)^5 \cdot 49 = (-1)(49) = -49$

$C_{33} = (-1)^{3+3} \cdot M_{33} = (-1)^6 \cdot (-8) = 1(-8) = -8$

Then the Matrix of co-factors is obtained as

$$\text{co-factor}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -54 & -41 & 12 \\ -40 & 29 & -42 \\ 36 & -49 & -8 \end{bmatrix}$$

Shortcut :- (To find co-factor of order 3)

$$\begin{bmatrix} -54 & 41 & 12 \\ 40 & 29 & 42 \\ 36 & 49 & -8 \end{bmatrix} \sim \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \sim \begin{bmatrix} -54 & -41 & 12 \\ -40 & 29 & -42 \\ 36 & -49 & -8 \end{bmatrix}$$

(Matrix of Minors) (Matrix of co-factors)

(Just apply the checkerboard of minuses to the Matrix of Minors)

$\text{Adjoint} = [\text{co-factor}(A)]^T$

So if we are going to find adjoint for the same question, then

As we already have the co-factor Matrix.

i.e.

$$\text{co-factor}(A) = \begin{bmatrix} -54 & -41 & 12 \\ -40 & 29 & -42 \\ 36 & -49 & -8 \end{bmatrix}$$

Then $Adj(A) = \left[co\text{-}factor(A)\right]^T$

$$= \begin{bmatrix} -54 & -40 & 36 \\ -41 & 29 & -49 \\ 12 & -42 & -8 \end{bmatrix}$$

Short-cut to find adjoint of a order two Matrix :-

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $Adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

i.e. Just swap the positions of a and d & put negative infront of b and c.

Conclusion :-

So by using adjoint of a Matrix and its determinant we can find inverse of the Matrix using formula

i.e. $\boxed{A^{-1} = \frac{Adj(A)}{|A|}}$, $|A| \neq 0$.

If $|A| = 0$, such a Matrix is called Singular Matrix.

→ Inverse of a Matrix A is A^{-1}

if $AA^{-1} = A^{-1}A = I$

where I is the Identity Matrix.

→ Inverse is needed as Matrix division is not possible.

e.g. we want to find matrix X and Matrix A and B are known.

Given $XA = B$.

But $X = B/A$ is not possible as we can't divide Matrices.

So we multiply both sides by A^{-1}

$\Rightarrow X A A^{-1} = B A^{-1}$

$\Rightarrow X I = B A^{-1} \quad (\because A A^{-1} = I)$

$\Rightarrow X = B A^{-1}$

By calculating A^{-1} & Multiplying it with B we can easily find Matrix X.

Working Rule ~~for~~ to calculate the Inverse.

- step1 – Calculate the determinant.
- step2 – Calculate the Matrix of minors.
- step3 – Turn that into Matrix of co-factors.
- step4 – find Adjoint Matrix from co-factor Matrix.
- step5 – Multiply that by $\frac{1}{\text{Determinant}}$.

Solution of System of linear Equations :-
(by Matrix Method)

System of linear equation means when we have two or more linear equations working together.

Suppose we have the following system of eq^{n}s.

$a_1 x + b_1 y + c_1 z = d_1$
$a_2 x + b_2 y + c_2 z = d_2$
$a_3 x + b_3 y + c_3 z = d_3$

Then the above system of eq^{n}s can be written in Matrix form i.e. $AX = B$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Known as co-efficient Matrix.

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ Matrix of unknowns and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

As we have to find solⁿ i.e. value of x, y, z, we need to find Matrix 'X'

as $AX = B$

$\Rightarrow X = BA^{-1}$

or $X = A^{-1}B$

$$= \frac{\text{Adj}(A)}{|A|} \cdot B$$

Q:-1 solve by Matrix Method.

$x + 2y = 3$, $3x + y = 4$

Solution:- Given linear eqⁿs are

$x + 2y = 3$

$3x + y = 4$

The above system of eqⁿs can be written in Matrix form i.e.

$$AX = B$$

where $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Then $X = A^{-1}B$ —— (1)

Now for A^{-1}

$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5 \neq 0$

$\Rightarrow A^{-1}$ exists.

$C_{11} = (-1)^{1+1} \cdot M_{11} = (-1)^2 \cdot 1 = 1$

$C_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^3 \cdot 3 = -3$

$C_{21} = (-1)^{2+1} \cdot M_{21} = (-1)^3 \cdot 2 = -2$

$C_{22} = (-1)^{2+2} \cdot M_{22} = (-1)^4 \cdot 1 = 1$

Then co-factor $(A) = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$

$$Adj(A) = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}}{-5}$$

putting these Values in eq^n ①

$$\Rightarrow X = A^{-1} B = \frac{Adj(A)}{|A|} \cdot B$$

$$\Rightarrow X = \frac{\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}}{-5}$$

$$\Rightarrow X = \frac{\begin{bmatrix} 3 + (-8) \\ -9 + 4 \end{bmatrix}}{-5}$$

$$\Rightarrow X = \begin{bmatrix} -5 \\ -5 \end{bmatrix} \times \frac{1}{-5}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

i.e. $x = 1$ & $y = 1$

Q.2 Solve $x - y + z = 4$

$2x + y - 3z = 0$

$x + y + z = 2$

Solution :- The above system of equations can be written in $AX = B$ form

where $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

Then $X = A^{-1} B$ —— ①

To find A^{-1}

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(1+3) + 1(2+3) + 1(2-1) = 4 + 5 + 1 = 10 \neq 0$$

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 1\left(1-(-3)\right) = 1+3 = 4$$

$$C_{12} = (-1)^{1+2} \cdot M_{12} = (-1)^3 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = (-1)\left(2-(-3)\right) = -1(5) = -5$$

$$C_{13} = (-1)^{1+3} \cdot M_{13} = (-1)^4 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (+1)(2-1) = 1$$

$$C_{21} = (-1)^{2+1} \cdot M_{21} = (-1)^3 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (-1)(-1-1) = (-1)(-2) = 2$$

$$C_{22} = (-1)^{2+2} \cdot M_{22} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1(1-1) = 0$$

$$C_{23} = (-1)^{2+3} \cdot M_{23} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = (-1)\left(1-(-1)\right) = (-1)(2) = -2$$

$$C_{31} = (-1)^{3+1} \cdot M_{31} = (-1)^4 \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 1(3-1) = 2$$

$$C_{32} = (-1)^{3+2} \cdot M_{32} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = (-1)(-3-2) = (-1)(-5) = 5$$

$$C_{33} = (-1)^{3+3} \cdot M_{33} = (-1)^6 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = (1)\left(1-(-2)\right) = 1+2 = 3$$

$$\text{co-factor}(A) = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

$$\text{Adj}(A) = \left[\text{co-factor}(A)\right]^T$$

$$= \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

putting these values in eqn (1)

$$X = A^{-1} B$$

$$= \frac{\text{Adj}(A)}{|A|} \cdot B$$

$$= \frac{\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}}{10}$$

$$= \begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix} \times \frac{1}{10}$$

$$= \begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} \times \frac{1}{10}$$

$$= \begin{bmatrix} 20/10 \\ -10/10 \\ 10/10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

So we have

$$X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\Rightarrow x = 2$, $y = -1$ and $z = 1$

which is the required solⁿ.

Important 5 Marks Questions:-

Q.1 Find the value of x and y

when $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Solution:- Given $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x + 3y \\ 2x - y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

i.e. $x + 3y = 4$ —— (I)

$2x - y = 1$ —— (II)

Solving (I) & (II)

eq (I) × 2 $\Rightarrow 2x + 6y = 8$

eq (II) × 1 $\Rightarrow 2x - y = 1$

$\quad (-) \quad (+) \quad (-)$

$7y = 7$

$\Rightarrow y = 7/7 = 1$

$\boxed{y = 1}$, putting value of $y = 1$ in

eqⁿ (I) we have $\boxed{x = 1}$

Q.2 Verify $(AB)^T = B^T A^T$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$$

Solution :- L.H.S

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+(-3) & 2+0+3 \\ 3+(-4)+(-1) & 6+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ -2 & 7 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & -2 \\ 5 & 7 \end{bmatrix}$$

R.H.S $\quad B^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix}$

$$B^T A^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+(-3) & 3+(-4)+(-1) \\ 2+0+3 & 6+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 5 & 7 \end{bmatrix}$$

L.H.S = R.H.S (proved).

Q.3 If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ and I is the 2×2 unit Matrix. Find $(A-2I)(A-3I)$.

Solution Given $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $A - 2I = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4-2 & 2-0 \\ -1-0 & 1-2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

Again $A - 3I$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 2-0 \\ -1-0 & 1-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

Now $(A-2I)(A-3I)$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+(-2) & 4+(-4) \\ -1+(1) & -2+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Q.4 solve by Matrix Method.

$$2x - y + 2 = 0$$
$$3x + 4y - 3 = 0$$

Solution:- Given linear equations are

$$2x - y = -2$$
$$3x + 4y = 3$$

The above system of linear eq^{n}s can be written in Matrix form.

i.e. $AX = B$

where $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Then $X = A^{-1} B$

$$= \frac{Adj(A)}{|A|} \cdot B$$

$$= \frac{\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}}{8-(-3)} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} \cdot \frac{1}{11}$$

$$= \begin{bmatrix} -5 \\ 12 \end{bmatrix} \times \frac{1}{11}$$

$$= \begin{bmatrix} -5/11 \\ 12/11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/11 \\ 12/11 \end{bmatrix}$$

So $x = -5/11$ & $y = 12/11$ is the required solution.

Q.5 If $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

Verify $A^2 - 3A + 2I = 0$

Solution $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

Then $A^2 = A \cdot A$.

$$= \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & -2+0+(-4) \\ 2+4+0 & 0+4+0 & -4+8+8 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix}$$

$$3A = 3 \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -6 \\ 6 & 6 & 12 \\ 0 & 0 & 6 \end{bmatrix}$$

$$2I = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Considere $\underline{L \cdot H \cdot S}$

$$A^2 - 3A + 2I$$

$$= \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -6 \\ 6 & 6 & 12 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+2 & 0-0+0 & -6+6+0 \\ 6-6+0 & 4-6+2 & 12-12+0 \\ 0-0+0 & 0-0+0 & 4-6+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R \cdot H \cdot S$$

(proved).

Properties of determinant :-

① Property of Reflection

The determinant remains unaltered if its rows are changed into columns and the columns into rows.

e.g:- $\Delta = \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} = 12 - 10 = 2$

using Property, $\Delta' = \begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} = 12 - 10 = 2$

② All-Zero property

If all the elements of a row or column are zero then the determinant is zero.

e.g:- $\Delta = \begin{vmatrix} 4 & 5 & 0 \\ 2 & -3 & 0 \\ 9 & 4 & 0 \end{vmatrix} = 0$

③ Proportionality (Repetition) Property

If all the elements of a row (column) are proportional (identical) to the elements of

Same other row (or column), then the determinant is zero. e.g. $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & -8 \\ 1 & 2 & 3 \end{vmatrix} = 0$ (as R_1 is identical to R_3)

(4) Switching Property

The interchange of any two rows (or columns), of a determinant changes its sign.

$$\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 2 & 0 & 3 \\ 4 & 1 & 0 \end{vmatrix}$$

$$= 4\begin{vmatrix} -1 & 4 \\ 0 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= 4(-3-0) - 1(6-8)$$

$$= -12-(-2) = -12+2 = -10$$

By using Property $R_1 \longleftrightarrow R_2$

$$\Delta' = \begin{vmatrix} 2 & 0 & 3 \\ 3 & -1 & 4 \\ 4 & 1 & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} -1 & 4 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix}$$

$$= 2(0-4) + 3(3+4)$$

$$= -8+18 = 10$$

(5) Scalar Multiple Property

If all the elements of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

e.g. $\Delta = \begin{vmatrix} 4 & 2 \\ -1 & 3 \end{vmatrix}$

$$= 12-(-2) = 14$$

By using Property $C_1 \to 2C_1$

$$\Delta' = \begin{vmatrix} 8 & 2 \\ -2 & 3 \end{vmatrix}$$

$$= 24-(-4)$$

$$= 28$$

(6) Sum property:

If each element in any row or column consists of two or more terms, then the determinant can be expressed as the sum of two or more than two determinants.

e.g. $\Delta = \begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$

(I) Property of Invariance :-

If the elements of any row (column) be increased or decreased by the same scalar with the corresponding elements of another row (or column) then the determinant remains unaltered.

e.g.

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 1(0-3) - 2(8-3) - 1(4-0)$$

$$= -3 - 10 - 4 = -17$$

using property

$$R_1 \to R_1 + 2R_3$$

$$\Delta' = \begin{vmatrix} 3 & 4 & 3 \\ 4 & 0 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} - 4\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 3(0-3) - 4(8-3) + 3(4-0) = -9 - 20 + 12$$
$$= -29 + 12$$
$$= -17$$

Q:-1 Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Proof :- L.H.S

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$C_1 \to C_1 - C_2 \quad \& \quad C_2 \to C_2 - C_3$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a+b) & (b-c)(b+c) & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\left\{1\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}\right\}$$

$$= (a-b)(b-c)\{(b+c)-(a+b)\}$$

$$= (a-b)(b-c)\{b+c-a-b\}$$

$$= (a-b)(b-c)(c-a) = \text{R.H.S (proved)}$$

(2) Prove that $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

Proof :- L.H.S

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$$

$C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} x & 0 & 1 \\ -y & y & 1 \\ 0 & -z & 1+z \end{vmatrix}$$

$$= \begin{vmatrix} x & 0 & 1 \\ -y & y & 1 \\ 0 & -z & 1+z \end{vmatrix}$$

Expand through R_1

$$= x\begin{vmatrix} y & 1 \\ -z & 1+z \end{vmatrix} + 1\begin{vmatrix} -y & y \\ 0 & -z \end{vmatrix}$$

$$= x\{(y)(1+z) - (-z)(1)\} + 1((-y)(-z) - 0)$$

$$= x\{y + yz + z\} + yz$$

$$= xy + xyz + xz + yz$$

$$= xyz\left(\frac{1}{z} + 1 + \frac{1}{y} + \frac{1}{x}\right)$$

$$= xyz\left(1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \text{R.H.S (proved)}$$

Method

Step-1 Try to Make factors. then take it common.

Step-2 Try to Make Zeros (as much as you can)

Step-3 Expand.

Q.3 Prove that

$$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$$

Proof

L.H.S

$$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} x+a+b+c & b & c \\ a+x+b+c & x+b & c \\ a+b+x+c & b & x+c \end{vmatrix}$$

Taking $(x+a+b+c)$ common from C_1

$$= (x+a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix}$$

$R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$$= (x+a+b+c) \begin{vmatrix} 0 & -x & 0 \\ 0 & x & -x \\ 1 & b & x+c \end{vmatrix}$$

$R_2 \to R_1 + R_2$

$$= (x+a+b+c) \begin{vmatrix} 0 & -x & 0 \\ 0 & 0 & -x \\ 1 & b & x+c \end{vmatrix}$$

Expand by taking C_1

$$= (x+a+b+c) \left\{ 1 \begin{vmatrix} -x & 0 \\ 0 & -x \end{vmatrix} \right\}$$

$= (x+a+b+c)(x^2 - 0)$

$= x^2(x+a+b+c) =$ R.H.S

(proved).

Q.4 Prove that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Proof :-

L.H.S

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$$

$C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \\ bc-ca & ca-ab & ab \end{vmatrix}$$

$$= \begin{vmatrix} a-b & b-c & c \\ (a+b)(a-b) & (b+c)(b-c) & c^2 \\ -c(a-b) & -a(b-c) & ab \end{vmatrix}$$

= Taking $(a-b)$ common from C_1 and $(b-c)$ common from C_2

$$= (a-b)(b-c)\begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -c & -a & ab \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & c \\ a-c & b+c & c^2 \\ a-c & -a & ab \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & c \\ -(c-a) & b+c & c^2 \\ -(c-a) & -a & ab \end{vmatrix}$$

Taking $(c-a)$ common from C_1

$$= (a-b)(b-c)(c-a) \begin{vmatrix} 0 & 1 & c \\ -1 & b+c & c^2 \\ -1 & -a & ab \end{vmatrix}$$

$\cancel{R_2 \to R_2 + R_3}$ $\quad R_2 \to R_2 - R_3$

$$= (a-b)(b-c)(c-a) \begin{vmatrix} 0 & 1 & c \\ 0 & b+c+a & c^2-ab \\ -1 & -a & ab \end{vmatrix}$$

Expand by taking (choosing) C_1

$$= (a-b)(b-c)(c-a) \left\{ -1 \begin{vmatrix} 1 & c \\ b+c+a & c^2-ab \end{vmatrix} \right\}$$

$$= (a-b)(b-c)(c-a) \left\{ -1 \left(\cancel{c^2} - ab - bc - \cancel{c^2} - ac \right) \right\}$$

$$= (a-b)(b-c)(c-a)(ab+bc+ca) = R.H.S \text{ (proved)}$$

Q:-5 Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Proof :- L.H.S

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ a+b+c & -(a+b+c) & 2b \\ 0 & a+b+c & c-a-b \end{vmatrix}$$

Expand by choosing R_1

$$= (a+b+c)\left\{ 1 \begin{vmatrix} a+b+c & -(a+b+c) \\ 0 & a+b+c \end{vmatrix} \right\}$$

$$= (a+b+c)\left\{ (a+b+c)^2 - 0 \right\}$$

$$= (a+b+c)^3 = \text{R.H.S (proved)}.$$

Q.6:- If $A+B+C=\pi$, P.T. $\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} = 0$.

L.H.S :-

$$\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$= \begin{vmatrix} \sin^2 A - \sin^2 B & \cot A - \cot B & 0 \\ \sin^2 B - \sin^2 C & \cot B - \cot C & 0 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$$

Expand by choosing C_3

$$= 1 \begin{vmatrix} \sin^2 A - \sin^2 B & \cot A - \cot B \\ \sin^2 B - \sin^2 C & \cot B - \cot C \end{vmatrix}$$

$$= (\sin^2 A - \sin^2 B)(\cot B - \cot C) - (\sin^2 B - \sin^2 C)(\cot A - \cot B)$$

$$= \{\sin(A+B)\cdot\sin(A-B)\}\left\{\frac{\cos B}{\sin B} - \frac{\cos C}{\sin C}\right\} - \{\sin(B+C)\cdot\sin(B-C)\}\left\{\frac{\cos A}{\sin A} - \frac{\cos B}{\sin B}\right\}$$

$$= \sin(A+B)\cdot\sin(A-B)\left\{\frac{\cos B\cdot\sin C - \cos C\cdot\sin B}{\sin B\cdot\sin C}\right\} - \sin(B+C)\cdot\sin(B-C)\left\{\frac{\cos A\cdot\sin B - \cos B\sin A}{\sin A\cdot\sin B}\right\}$$

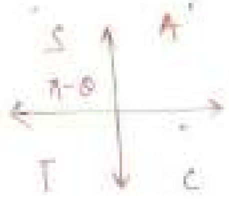
$$= \frac{\sin(A+B)\sin(A-B)\cdot\sin(C-B)}{\sin B\cdot\sin C} - \frac{\sin(B+C)\cdot\sin(B-C)\cdot\sin(B-A)}{\sin A\cdot\sin B}.$$

$A+B+C=\pi$

$\Rightarrow A+B=\pi-C$

$\Rightarrow \sin(A+B)=\sin(\pi-C)$

$=\sin C$



Similarly $\sin(B+C)=\sin A$

$$= \frac{\cancel{\sin C}\cdot\sin(A-B)\cdot\sin(C-B)}{\sin B\cdot\cancel{\sin C}} - \frac{\cancel{\sin A}\cdot\sin(B-A)\cdot\sin(B-C)}{\sin B\cdot\cancel{\sin A}}$$

$$= \frac{\sin(A-B)\cdot\sin(C-B)}{\sin B} - \frac{\sin(B-A)\cdot\sin(B-C)}{\sin B}$$

$$= \frac{\sin\{-(B-A)\}\cdot\sin\{-(B-C)\}}{\sin B} - \frac{\sin(B-A)\cdot\sin(B-C)}{\sin B}$$

$(\because \sin(-\theta)=-\sin\theta)$

$$= \frac{-\sin(B-A)\cdot-\sin(B-C)}{\sin B} - \frac{\sin(B-A)\cdot\sin(B-C)}{\sin B}$$

$$= \frac{\sin(B-A)\cdot\sin(B-C)}{\sin B} - \frac{\sin(B-A)\cdot\sin(B-C)}{\sin B}$$

$= 0 =$ R.H.S. (proved)

Solution of Linear System of Equations using Determinant :-

Let's consider the following System of linear equations.

i.e. $a_1x+b_1y=c_1$ —— (I)

$a_2x+b_2y=c_2$ —— (II)

To solve 'x'

eqn (I) $\times b_2 \Rightarrow b_2a_1x+\cancel{b_2b_1y}=b_2c_1$

eqn (II) $\times b_1 \Rightarrow b_1a_2x+\cancel{b_1b_2y}=b_1c_2$

(−) (−) (−)

$\Rightarrow b_2a_1x-b_1a_2x=b_2c_1-b_1c_2$

$\Rightarrow (b_2a_1-b_1a_2)x=b_2c_1-b_1c_2$

$\Rightarrow x=\dfrac{b_2c_1-b_1c_2}{b_2a_1-b_1a_2}$

To solve 'y'

eqn (I) $\times a_2 \Rightarrow \cancel{a_2a_1x}+a_2b_1y=a_2c_1$

eqn (II) $\times a_1 \Rightarrow \cancel{a_1a_2x}+a_1b_2y=a_1c_2$

(−) (−) (−)

$\Rightarrow (a_2b_1-a_1b_2)y=a_2c_1-a_1c_2$

$$\Rightarrow y = \frac{a_2 c_1 - a_1 c_2}{a_2 b_1 - a_1 b_2}$$

So the required solutions are

$$x = \frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1} \quad , \quad y = \frac{a_2 c_1 - a_1 c_2}{a_2 b_1 - a_1 b_2}$$

We get the above solution by using elimination method, but the above can be represented ~~by~~ ~~it~~ in the form of determinant.

$$\Rightarrow x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \& \quad y = \frac{-(a_1 c_2 - a_2 c_1)}{-(a_1 b_2 - a_2 b_1)}$$

$$= \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 c_1}$$

$$= \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

~~bathara~~ So this is the solⁿ we are getting by using ~~cramer~~ determinant, is known as Cramer's Rule

So Cramer's Rule is a Method that uses determinants to solve system of linear eqⁿs. which introduces new notations also.

→ we can notice that the denominator of both x & y is the determinant of coefficient Matrix.

and Δ = Determinant of the coefficient Matrix.

Δ_x = Determinant of coefficient Matrix in which the x-column is replaced by the constant column.

* and this is the numerator in the solution of 'x'.

Δ_y = Determinant of coefficient Matrix in which the y-column is replaced by the constant column.

* and this the numerator in the solution of y.

So finally we can say

$$\boxed{x = \frac{\Delta_x}{\Delta}} \quad , \quad \boxed{y = \frac{\Delta_y}{\Delta}}$$

Q.1 Solve $2x-y=2$
$3x+y=13$ using cramer's rule.

Solution Given $2x-y=2$
$3x+y=13$

Then $\Delta = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2-(-3) = 2+3 = 5$

$\Delta_x = \begin{vmatrix} 2 & -1 \\ 13 & 1 \end{vmatrix} = 2-(-13) = 15$

$\Delta y = \begin{vmatrix} 2 & 2 \\ 3 & 13 \end{vmatrix} = 26-6 = 20$

So $x = \frac{\Delta x}{\Delta} = \frac{15}{5} = 3$

$y = \frac{\Delta y}{\Delta} = \frac{20}{5} = 4$

So the required solⁿ. is $x=3, y=4$

Q.2 Solve $x+y+z=3$
$2x+3y+4z=9$
$x+2y-4z=-1$ using cramer's Rule.

Solution Given $x+y+z=3$
$2x+3y+4z=9$
$x+2y-4z=-1$

$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix}$

$= 1\begin{vmatrix} 3 & 4 \\ 2 & -4 \end{vmatrix} - 1\begin{vmatrix} 2 & 4 \\ 1 & -4 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$

$= 1(-12-8) - 1(-8-4) + 1(4-3)$

$= -20 - 1(-12) + 1$

$= -20+12+1 = -7$

$\Delta_x = \begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 4 \\ -1 & 2 & -4 \end{vmatrix}$

$= 3\begin{vmatrix} 3 & 4 \\ 2 & -4 \end{vmatrix} - 1\begin{vmatrix} 9 & 4 \\ -1 & -4 \end{vmatrix} + 1\begin{vmatrix} 9 & 3 \\ -1 & 2 \end{vmatrix}$

$$= 3\,(-12-8) - 1\,(-36+4) + 1\,(18-(-3))$$

$$= -60 + 32 + 21$$

$$= -7$$

$$\Delta_y = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 9 & 4 \\ 1 & -1 & -4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 9 & 4 \\ -1 & -4 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 1 & -4 \end{vmatrix} + 1\begin{vmatrix} 2 & 9 \\ 1 & -1 \end{vmatrix}$$

$$= 1\,(-36+4) - 3\,(-8-4) + 1\,(-2-9)$$

$$= -32 + 36 - 11$$

$$= -7$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 9 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & 9 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= (-3-18) - 1\,(-2-9) + 3\,(4-3)$$

$$= (-21) + 11 + 3$$

$$= -7$$

So $x = \dfrac{\Delta_x}{\Delta} = \dfrac{-7}{-7} = 1$

$y = \dfrac{\Delta_y}{\Delta} = \dfrac{-7}{-7} = 1$

$z = \dfrac{\Delta_z}{\Delta} = \dfrac{-7}{-7} = 1$.

So the required sol^n is $x=1$, $y=1$ & $z=1$.

TRIGONOMETRY

→ The word trigonometry comes from three greek words.

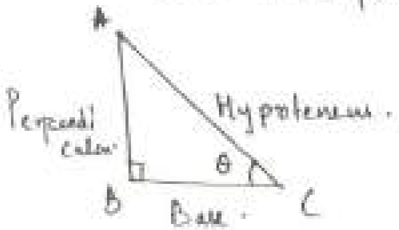
TRI → Three

GON → Sides

METRON – Measurement.

So Trigonometry means ~~meant~~ measurement of three sides of basically triangle (i.e. right angled triangle).

NOTE :– ① There are three Sides of triangle namely Base, Perpendicular and Hypotenuse.



i.e. Longest side = Hypotenuse (H)

Side infront of angle θ – perpendicular (P)

Remaining ~~one~~ side = Base (B)

② If B, P, H are three sides of right angled triangle.

Then $\boxed{P^2 + B^2 = H^2}$ Known as pythagoras theorem.

③ By taking ratios of sides we have.

$\frac{P}{H}, \frac{B}{H}, \frac{H}{P}, \frac{H}{B}, \frac{P}{B}, \frac{B}{P}$ (i.e. 6 ratios)

Known as trigonometric ratios.

In mathematics the above ratios have some specific names as follows:

Trigonometric Ratios

$\sin\theta = \frac{P}{H}$	$\operatorname{cosec}\theta = \frac{H}{P}$
$\cos\theta = \frac{B}{H}$	$\sec\theta = \frac{H}{B}$
$\tan\theta = \frac{P}{B}$	$\cot\theta = \frac{B}{P}$

from the above trigonometric Ratios.
We can derive the followings:

① $\sin\theta = \frac{1}{\operatorname{cosec}\theta}$

or $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

② $\cos\theta = \frac{1}{\sec\theta}$

or $\sec\theta = \frac{1}{\cos\theta}$

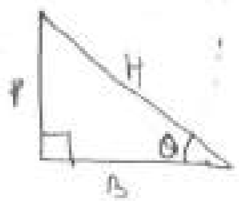
③ $\tan\theta = \frac{1}{\cot\theta}$

or $\cot\theta = \frac{1}{\tan\theta}$

④ $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

⑤ Pythagoras Theorem:-



$P^2 + B^2 = H^2$

$\Rightarrow \frac{P^2}{H^2} + \frac{B^2}{H^2} = 1$

$\Rightarrow \left(\frac{P}{H}\right)^2 + \left(\frac{B}{H}\right)^2 = 1$

$\Rightarrow (\sin\theta)^2 + (\cos\theta)^2 = 1$

$\Rightarrow \boxed{\sin^2\theta + \cos^2\theta = 1}$

or $\sin^2\theta = 1 - \cos^2\theta$

$\cos^2\theta = 1 - \sin^2\theta$

⑥ Dividing $\sin^2\theta$ in formula ⑤

$\boxed{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$

⑦ Dividing $\cos^2\theta$ in ⑤

$\boxed{\tan^2\theta + 1 = \sec^2\theta}$

Angles :-

→ Angles basically denoted by $\theta, \alpha, \beta, \gamma$

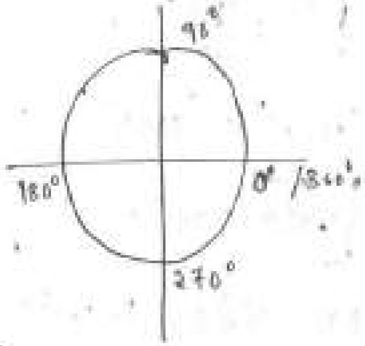
measured in anti-clockwise direction.

If measured in clockwise direction the angle will be negative angle.

→ There are two units to measure angle.

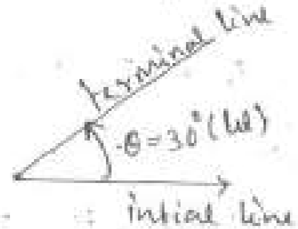
(i) Degree (ii) Radian

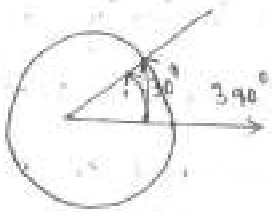
Degree	Radian
0°	0
30°	$\pi/6$
45°	$\pi/4$
60°	$\pi/3$
90°	$\pi/2$
180°	π
270°	$3\pi/2$
360°	2π



Co-terminal angles:-

If the terminal lines of two angles are same then, the angles are known as co-terminal angles.





$\theta = 30^\circ$

$30^\circ + 360^\circ = 390^\circ$ — are co-terminal angles.

$390^\circ + 360^\circ = 750^\circ$

$\theta = 30^\circ$

$30^\circ - 360^\circ = -330^\circ$ — co-terminal angles.

$-330^\circ - 360^\circ = -690^\circ$

Trigonometric Ratios of some standard angles.

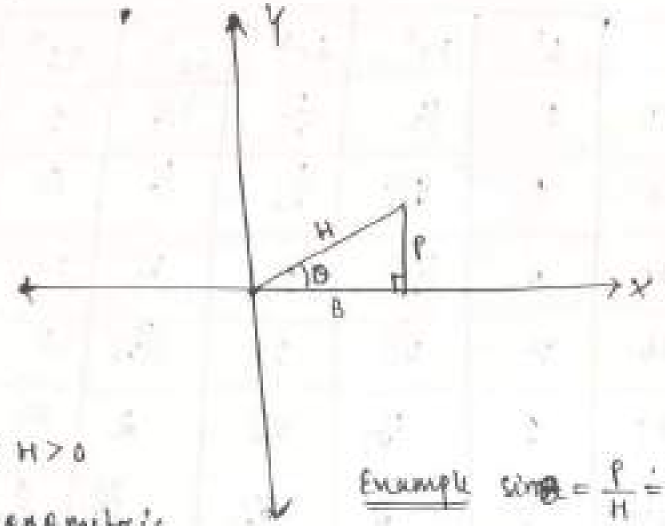
	0°	30°	45°	60°	90°
sin	0	$1/2$	$\frac{1}{\sqrt{2}}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$\frac{1}{\sqrt{2}}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0
sec	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞
cosec	∞	2	$\sqrt{2}$	$2/\sqrt{3}$	1

Trigonometric functions:-

There are 6 trigonometric functions

sinx, cosx, tanx, cotx, secx and cosecx

which are represented in two dimension (plane.



1st quadrant

B > 0, P > 0 & H > 0

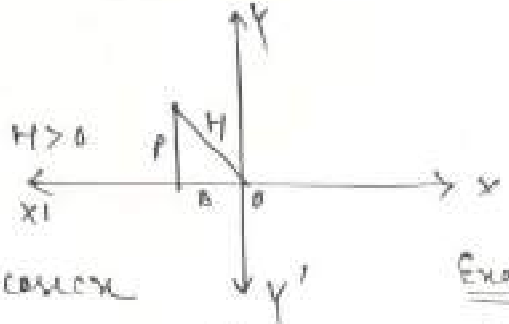
So all the trigonometric functions are +ve.

Example $\sin\theta = \frac{P}{H} = \frac{+}{+} = +ve$

$\sec\theta = \frac{H}{B} = \frac{+ve}{+ve} = +ve$

2nd Quadrant

B < 0, P > 0 & H > 0



So sin and cosecx are +ve and other functions are -ve.

Example

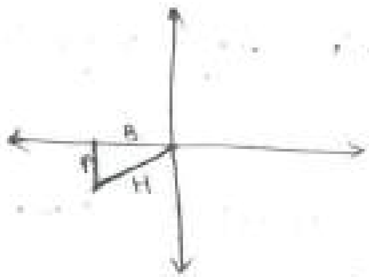
$\sin\theta = \frac{P}{H} = \frac{+ve}{+ve} = +ve$

$\tan\theta = \frac{P}{B} = \frac{+ve}{-ve} = -ve$

3rd Quadrant

In 3rd quadrant

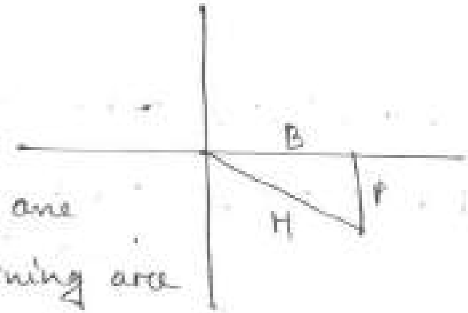
B < 0, P < 0 & H > 0



So tanθ and cotθ are +ve and remaining functions are -ve.

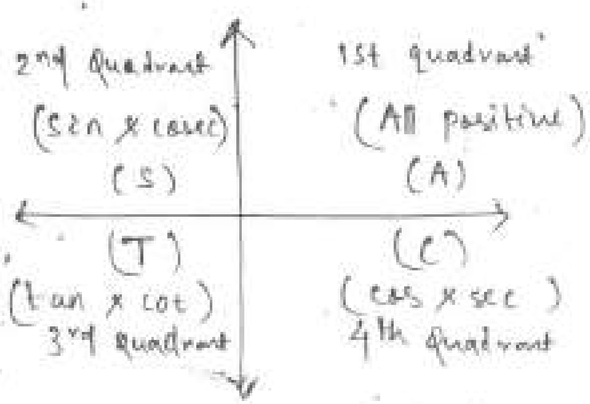
4th Quadrant

P < 0, B > 0, H > 0



So cosθ & secθ are +ve and remaining are -ve.

So finally.



A S T C Rule.

Conversion of any angle into acute angle :-

Step 1 convert the given angle into the following. format.

$90+\theta$ $180^\circ-\theta$ $270^\circ-\theta$ $360^\circ-\theta$

$90^\circ-\theta$ $180^\circ+\theta$ $270^\circ+\theta$ $360^\circ+\theta$

Step 2 for $180^\circ/360^\circ \Rightarrow$ ~~ratios~~ function same.

for $90^\circ/270^\circ \Rightarrow$ $\sin \longleftrightarrow \cos$

$\tan \longleftrightarrow \cot$

$\sec \longleftrightarrow \operatorname{cosec}$

Step 3 Then put the sign depending upon the original function.

Example $\sin 210^\circ$

$= \sin(180^\circ + 30^\circ)$ ($\rightarrow 180^\circ+30^\circ$ will lie on 3^{rd} Quadrant. so put (-ve) sign as sin function is -ve on 3^{rd} Quadrant)

$= -\sin 30^\circ$

$= -\frac{1}{2}$

Domain and Range of Trigonometric functions

① $f(x) = \sin x$

Here x can be any angle $-\infty$ to ∞

so Domain = R

$x \in R$; $\sin x = \frac{P}{H}$ where $H \geq P$ and

then Range will be $[-1, 1]$

② $f(x) = \cos x$

Here x can be any angle $-\infty$ to ∞

So Domain = R

$\cos x = \frac{B}{H}$ where $H \geq B$

then Range will be $[-1, 1]$

③ $f(x) = \tan x = \frac{\sin x}{\cos x}$, $\cos x \neq 0$

i.e. $x \neq \pi/2, 3\pi/2, 5\pi/2 \cdots$

i.e. $x \neq (2n+1)\pi/2$

Domain $= R - \{(2n+1)\pi/2, n \in Z\}$

$\tan x = \frac{P}{B}$ so Range = R

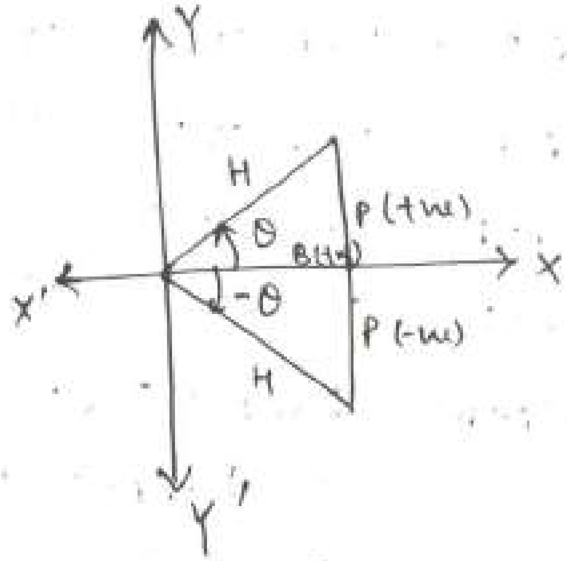
Function	Domain	Range
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$R - \{(2n+1)\pi/2, n \in Z\}$	R
$\cot x$	$R - \{n\pi, n \in Z\}$	R
$\sec x$	$R - \{(2n+1)\pi/2, n \in Z\}$	$(-\infty, -1] \cup [1, \infty)$
$\operatorname{cosec} x$	$R - \{n\pi, n \in Z\}$	$(-\infty, -1] \cup [1, \infty)$

Trigonometric Ratios of negative angles:-

① $\sin(-\theta) = \frac{-P}{H} = \cancel{\frac{\quad}{+ve}}$

$= -\left(\frac{P}{H}\right)$

$= -\sin\theta$



② $\cos(-\theta) = \frac{B}{H} = \cos\theta$

similarly

③ $\tan(-\theta) = -\tan\theta$

④ $\cot(-\theta) = -\cot\theta$

⑤ $\sec(-\theta) = \sec\theta$

⑥ $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$.

NOTE :- cos & sec are even trigonometric functions. and others are odd functions.

Q:-1 $\tan(-840^\circ)$

$= -\tan 840^\circ = -\tan\{120^\circ\}$ (using co-terminal angle.

$= -\tan(90^\circ + 30^\circ)$

$= -[-\cot 30^\circ] = \cot 30^\circ = \sqrt{3}$

Trigonometric functions of compound angles :-

1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
2. $\sin(A-B) = \sin A \cos B - \cos A \cdot \sin B$
3. $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$
4. $\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$
5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$
7. $\cot(A+B) = \dfrac{\cot A \cdot \cot B - 1}{\cot B + \cot A}$
8. $\cot(A-B) = \dfrac{\cot A \cdot \cot B + 1}{\cot B - \cot A}$

Q:- find $\cos 15^\circ$

Sol^n $\cos 15^\circ = \cos(45^\circ - 30^\circ)$

$= \cos 45^\circ \cdot \cos 30^\circ + \sin 45^\circ \cdot \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

Trigonometric formulas of multiple angles :-

1. $\sin 2A = 2 \sin A \cdot \cos A$

   or $\dfrac{2 \tan A}{1 + \tan^2 A}$

2. $\cos 2A = \cos^2 A - \sin^2 A$

   or $2\cos^2 A - 1$

   or $1 - 2\sin^2 A$

   or $\dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

3. $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

4. $\sin 3A = 3 \sin A - 4 \sin^3 A$

5. $\cos 3A = 4 \cos^3 A - 3 \cos A$

6. $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$.

Proof ① $\sin 2A = \sin(A+A) = \sin A \cos A + \cos A \sin A$

$= 2 \sin A \cos A$.

or $\sin 2A = \dfrac{2 \sin A \cdot \cos A}{\cos^2 A} \cdot \cos^2 A$

$= \dfrac{2 \tan A}{\sec^2 A} = \dfrac{2 \tan A}{1 + \tan^2 A}$

Proof (2) $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

$$= \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

Q:- Evaluate $\sin 18^\circ$

$$A = 18^\circ$$
$$\Rightarrow 5A = 90^\circ$$
$$\Rightarrow 2A + 3A = 90^\circ$$
$$\Rightarrow 2A = 90^\circ - 3A$$
$$\Rightarrow \sin 2A = \sin(90^\circ - 3A) = \cos 3A$$
$$\Rightarrow 2\sin A \cos A = 4\cos^3 A - 3\cos A$$
$$\Rightarrow 2\sin A = 4\cos^2 A - 3 = 4(1-\sin^2 A) - 3$$
$$= 4 - 4\sin^2 A - 3$$
$$\Rightarrow 4\sin^2 A + 2\sin A - 1 = 0$$
$$\sin A = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{\pm\sqrt{5} - 1}{4}$$

as 18° lies on 1st Quadrant so ~~angle its~~ value will be positive.

$$\sin 18^\circ = \frac{\sqrt{5} - 1}{4}$$

Trigonometric Formulas of Submultiple Angles

① $\sin A = 2\sin\frac{A}{2} \cdot \cos\frac{A}{2}$ or $\dfrac{2\tan A/2}{1+\tan^2 A/2}$

② $\cos A = \cos^2 A/2 - \sin^2 A/2$

or $2\cos^2 A/2 - 1$

or $1 - 2\sin^2 A/2$

or $\dfrac{1-\tan^2 A/2}{1+\tan^2 A/2}$

③ $\tan A = \dfrac{2\tan A/2}{1-\tan^2 A/2}$

④ $\sin^2 A/2 + \cos^2 A/2 = 1$

Q:- Prove that $\cot 7\frac{1}{2}^\circ = \sqrt{6}+\sqrt{3}+\sqrt{2}+2$

Proof $\cot \theta = \dfrac{1+\cos 2\theta}{\sin 2\theta}$

$\Rightarrow \cot \frac{\theta}{2} = \dfrac{1+\cos\theta}{\sin\theta}$

put $\theta = 15^\circ$

$\Rightarrow \cot 7\frac{1}{2}^\circ = \dfrac{1+\cos 15^\circ}{\sin 15^\circ}$

$= \dfrac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$

$= \dfrac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}$

$= \dfrac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \dfrac{2\sqrt{6}+2\sqrt{3}+2\sqrt{2}+4}{2}$

$= \sqrt{6}+\sqrt{3}+\sqrt{2}+2$ (proved).

Some Special formulas:

① $\sin(A+B)+\sin(A-B) = 2\sin A\cdot\cos B$

② $\sin(A+B)-\sin(A-B) = 2\cos A\cdot\sin B$

③ $\cos(A+B)+\cos(A-B) = 2\cos A\cdot\cos B$

④ $\cos(A+B)-\cos(A-B) = -2\sin A\cdot\sin B$

Put $A = \frac{C+D}{2}$, $B = \frac{C-D}{2}$; Then $A+B = C$
& $A-B = D$

⑤ $\sin C + \sin D = 2\sin\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$

⑥ $\sin C - \sin D = 2\cos\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$

⑦ $\cos C + \cos D = 2\cos\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$

⑧ $\cos C - \cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$

Two-Dimensional Geometry :-

(Co-ordinate Geometry)

Introduction :-

→ Co-ordinate Geometry is a link between the geometry and algebra, in which the geometrical problems are solved through algebra using curves and lines.

→ It is a part of geometry, where the position of points is described using an ordered pair of numbers on the plane.

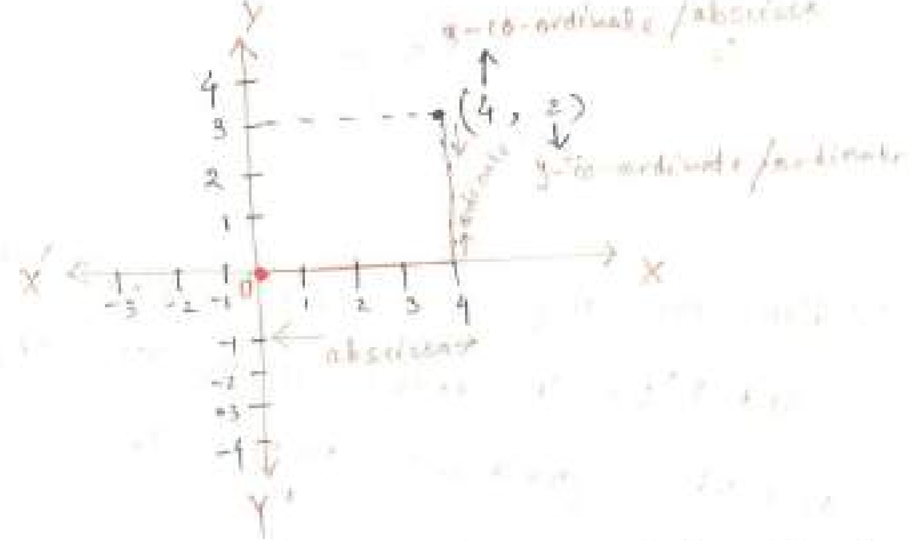
Application :- using co-ordinate geometry, it is possible to find the distance between two pts, to calculate area of a triangle in co-ordinate plane.

Co-ordinate :-

Co-ordinates are a set of values which helps to show the exact position of a pt. in the co-ordinate plane.

Co-ordinate plane :-

Co-ordinate plane is formed by intersection of two perpendicular lines X-axis & Y-axis, which is also known as cartesian plane is divided into four quadrants by the two axes.

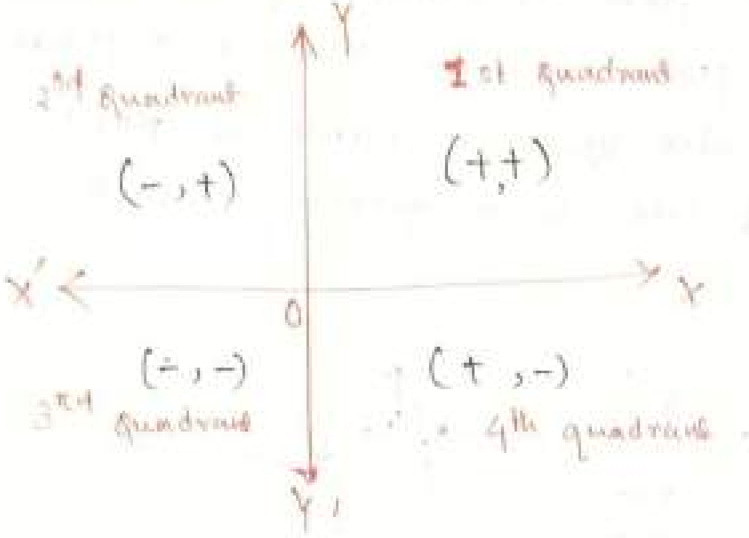


* The point at which the axes intersect is known as the origin.

* The location of any pt. on a plane is expressed by a pair of values (x, y), known as the co-ordinates.

* The horizontal line (X-axis) & vertical line (Y-axis) are known as co-ordinate axes.

Quadrants :-



→ Here the regions XOY, YOX', X'OY' and Y'OX are known as the 1st, 2nd, 3rd and 4th quadrant respectively.

→ Signs for a point in different quadrants are given as follows:

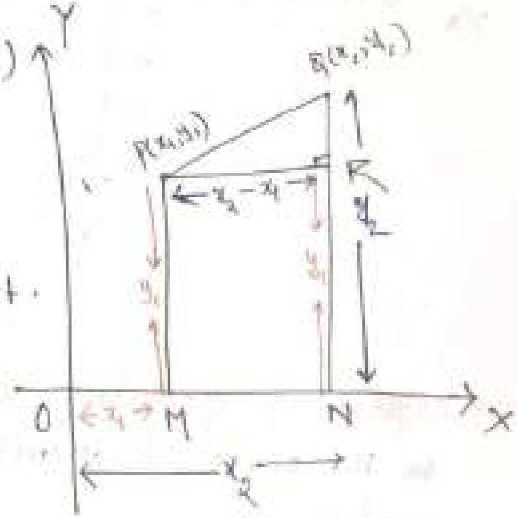
- 1st quadrant → $(+x, +y)$
- 2nd quadrant → $(-x, +y)$
- 3rd quadrant → $(-x, -y)$
- 4th quadrant → $(+x, -y)$.

Co-ordinate Geometry formulae & Theorems :-

Theorem 1 :- The distance between two pts $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

→ Here let $P(x_1, y_1)$ & $Q(x_2, y_2)$ be two pts.



→ let's draw perpendiculars PM and QN from the pt. P and Q on X-axis.

→ Also draw $PR \perp QN$.

Then $OM = x_1$ $ON = x_2$
$PM = y_1$ $QN = y_2$

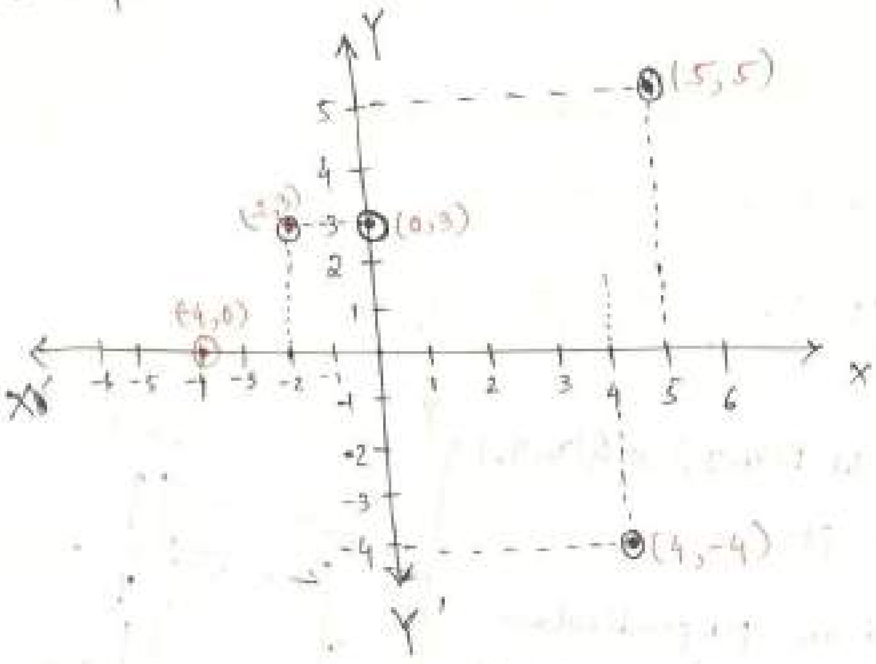
Then $PR = MN = ON - OM = x_2 - x_1$

Again $QR = QN - RN = y_2 - y_1$

Now consider the right angled triangle PQR.

Plotting of pts on cartesian plane :-

Let's plot (-2, 3), (4, -4), (5, 5), (0, 3), (-4, 0)



* System of geometry where the position of pts on the plane is described by using ordered pair of numbers.
* The plane where the pts are placed on, is known as co-ordinate plane.

  It has two dimensions.
* A pt's location on a plane is given by two numbers, 1^{st} tells where it is on X-axis & second where it is on Y-axis. & together they define a single & unique position on the plane.

By Pythagoras Theorem,

we have $(PQ)^2 = (PR)^2 + (QR)^2$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\Rightarrow \boxed{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Q.1 :- Find the distance between the pts (2, -4), (5, 6).

Solution :- Let $P(2, -4)$ and $Q(5, 6)$ are two given pts.

$P(2, -4)$, $x_1 = 2$, $y_1 = -4$

$Q(5, 6)$, $x_2 = 5$, $y_2 = 6$

Then $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(5-2)^2 + (6-(-4))^2}$$

$$= \sqrt{(3)^2 + (10)^2}$$

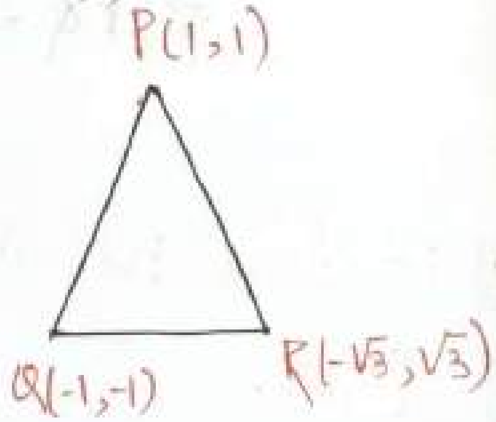
$$= \sqrt{9+100} = \sqrt{109}$$

Q:-2 Show that the pts $(1, 1)$, $(-1, -1)$ and $(-\sqrt{3}, \sqrt{3})$ are the vertices of an equilateral triangle.

Let $P(1, 1) \Rightarrow x_1 = 1,\ y_1 = 1$

$Q(-1, -1) \Rightarrow x_2 = -1,\ y_2 = -1$

$R(-\sqrt{3}, \sqrt{3}) \Rightarrow x_3 = -\sqrt{3},\ y_3 = \sqrt{3}$



are three given pts.

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-1-1)^2 + (-1-1)^2}$$

$$= \sqrt{(-2)^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8}$$

$$QR = \sqrt{\left(-\sqrt{3} - (-1)\right)^2 + \left(\sqrt{3} - (-1)\right)^2}$$

$$= \sqrt{(-\sqrt{3}+1)^2 + (\sqrt{3}+1)^2}$$

$$= \sqrt{(1)^2 + (\sqrt{3})^2 + 2(\sqrt{3})(1) + (\sqrt{3}) + (1)^2 + 2(\sqrt{3})(1)}$$

$$= \sqrt{1+3+3+1}$$

$$= \sqrt{8}$$

$PR = \sqrt{(-\sqrt{3}-1)^2 + (\sqrt{3}-1)^2}$

$= \sqrt{(-\sqrt{3})^2 + (-1)^2 + 2(-\sqrt{3})(-1) + (\sqrt{3})^2 + (1)^2 + 2\cdot(\sqrt{3})(-1)}$

$= \sqrt{3+1+3+1}$

$= \sqrt{8}$

Here $PQ = QR = PR$

$\Rightarrow$ PQR is an equilateral triangle.

Theorem-2 (Section formula).

Section formula helps in finding the co-ordinate of a pt. which divides a line segement in some ratio let $m:n$.

⇒ If $m=n$, then the pt. is the mid point.

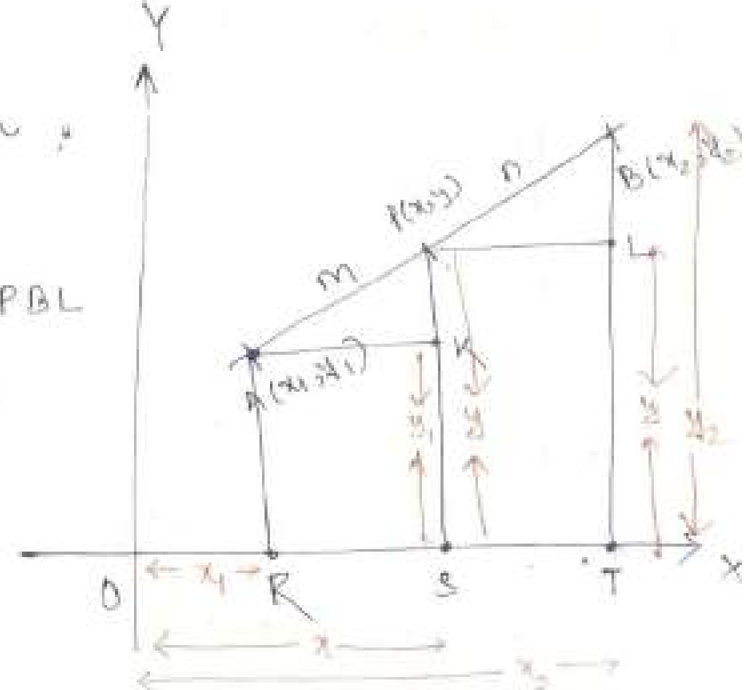
Internal divison with Section formula

Let $P(x,y)$ be a pt. which lies on a line-segment $\overline{AB}$ and satisfies $AP:PB = m:n$ then we can say P divides the line $\overline{AB}$ ~~in the r~~ internally in the ratio $m:n$.

Then co-ordinates of P will be

$$= \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n} \right)$$

In the given figure $\triangle APK$ & $\triangle PBL$ are similar.



⇒ Sides are proportional.

i.e. $\frac{AP}{PB} = \frac{AK}{PL} = \frac{PK}{BL}$

consider first two ratios.

$$\frac{AP}{PB} = \frac{AK}{PL}$$

$$\Rightarrow \frac{m}{n} = \frac{x-x_1}{x_2-x}$$

$$\Rightarrow m(x_2-x) = n(x-x_1)$$

$$\Rightarrow mx_2 - mx = nx - nx_1$$

$$\Rightarrow mx_2 + nx_1 = mx + nx$$

$$\Rightarrow mx_2 + nx_1 = x(m+n)$$

$$\Rightarrow \boxed{x = \frac{mx_2+nx_1}{m+n}}$$

Consider 1st and 3rd ratio.

$$\frac{AP}{PB} = \frac{PK}{BL}$$

$$\Rightarrow \frac{m}{n} = \frac{y - y_1}{y_2 - y}$$

$$\Rightarrow m y_2 - my = ny - n y_1$$

$$\Rightarrow m y_2 + n y_1 = ny + my$$

$$\Rightarrow m y_2 + n y_1 = (m+n) y$$

$$\Rightarrow \boxed{y = \frac{m y_2 + n y_1}{m+n}}$$

External Division with Section Formula

Let $P(x,y)$ be a pt. lies on the ~~outer~~ extention of the line segment AB and satisfies $AP : BP = m : n$

$\Rightarrow$ P divides the line segement externally in the ratio $m : n$

Then $x = \frac{m x_2 - n x_1}{m - n}$

$y = \frac{m y_2 - n y_1}{m - n}$

Q:-1 :- find the co-ordinates of the pt. which divides the line segment $(1, -3)$ and $(-3, 9)$ in the ratio $1 : 3$

Sol^n

let $A(1, -3)$ || $x_1 = 1$, $y_1 = -3$

and $B(-3, 9)$ || $x_2 = -3$, $y_2 = 9$

~~we~~ makes the line segement AB.

and $P(x,y)$ divides AB in the ratio $1 : 3$

so $m = 1$
$n = 3$

Then co-ordinates of P is obtained by

$$= \left(\frac{m x_2 + n x_1}{m+n}, \frac{m y_2 + n y_1}{m+n} \right)$$

$$= \left(\frac{1(-3) + 3(1)}{1+3}, \frac{1(9) + 3(-3)}{1+3} \right)$$

$$= \left(\frac{0}{4}, \frac{0}{4} \right) = (0, 0)$$

Q.2 Find the co-ordinate of the pt. which divides the line segment joining (1, -3) & (-3, 9) in the ratio 1:3.

Solution: Let $P(x, y)$ be the pt which divides the line segment joining $(1, -3)$ $\left\|\begin{matrix} x_1 = 1 \\ y_1 = -3 \end{matrix}\right.$

and $(-3, 9)$ $\left\|\begin{matrix} x_2 = -3 \\ y_2 = 9 \end{matrix}\right.$

in the ration 1 : 3

$\Rightarrow m = 1$

$n = 3$

Then $x = \dfrac{mx_2 + nx_1}{m+n} = \dfrac{1(-3) + 3(1)}{1+3}$

$= \dfrac{-3+3}{4} = \dfrac{0}{4} = 0$

$y = \dfrac{my_2 + ny_1}{m+n} = \dfrac{1(9) + 3(-3)}{1+3} = \dfrac{9-9}{4} = \dfrac{0}{4}$

$= 0$

So the required co-ordinate is (0, 0).

Q.3 Find the ratio in which the pt (3, -2) divide the line segment joining pts (1, 4) & (-3, 16).

Solution:- Let $P(3, -2)$ $\left\|\begin{matrix} x = 3 \\ y = -2 \end{matrix}\right.$

divides the line segment joining $(1, 4)$ $\left\|\begin{matrix} x_1 = 1 \\ y_1 = 4 \end{matrix}\right.$

and $(-3, 16)$ $\left\|\begin{matrix} x_2 = -3 \\ y_2 = 16 \end{matrix}\right.$

in the ratio m:n

Then $x = \dfrac{mx_2 + nx_1}{m+n}$

$\Rightarrow 3 = \dfrac{m(-3) + n(1)}{m+n}$

$\Rightarrow 3(m+n) = -3m + n$

$\Rightarrow 3m + 3n = -3m + n$

$\Rightarrow 6m = -2n$

$\Rightarrow \dfrac{m}{n} = \dfrac{-2}{6} = \dfrac{-1}{3}$

$\therefore$ P(3, -2) divides the line segment externally in the ratio 1 : 3.

~~② Externally →~~

Q.4 find the co-ordinate of the pt. which divides the line segment joining (2, −1) & (−3, 4) in the ratio 2 : 3 externally.

Solution :- let P(x, y) be the pt. which divides the line segment joining (2, −1) $\| \; x_1 = 2, \; y_1 = -1$

and (−3, 4) $\| \; x_2 = -3, \; y_2 = 4$

in the ratio 2 : 3 externally.

Then $x = \cancel{\dfrac{mx_2 + nx_1}{m+n}} \; \cancel{= \dfrac{m(-3)+n(2)}{m}}$

$\cancel{= \dfrac{2(-3)+3(2)}{2+3}} \; \cancel{= -6+}$

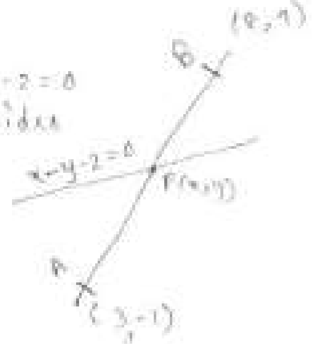
$$x = \frac{mx_2 - nx_1}{m-n} = \frac{2(-3) - 3(2)}{2-3} = \frac{-6-6}{-1} = \frac{-12}{-1} = 12$$

$$y = \frac{my_2 - ny_1}{m-n} = \frac{2(4) - 3(-1)}{2-3} = \frac{8+3}{-1} = \frac{11}{-1} = -11$$

So the required co-ordinate of pt. 'P' is (12, −11).

Q.5 find the ratio in which the line $x - y - 2 = 0$ cuts the line segment joining (3, −1) & (8, 9).

Solution



let P(x, y) be a pt on the line $x - y - 2 = 0$ which divides the line segment joining A(3, −1) and B(8, 9) in the ratio m : n.

$x_1 = 3, \; y_1 = -1$; $x_2 = 8, \; y_2 = 9$

Then $x = \dfrac{mx_2 + nx_1}{m+n}$ & $y = \dfrac{my_2 + ny_1}{m+n}$

$= \dfrac{8m + 3n}{m+n}$; $= \dfrac{9m - n}{m+n}$

As $P\left(\dfrac{8m+3n}{m+n}, \dfrac{9m-n}{m+n}\right)$ be a pt. on $x-y-2=0$

$$\Rightarrow \frac{8m+3n}{m+n} - \frac{9m-n}{m+n} - 2 = 0$$

$$\Rightarrow \frac{8m+3n-9m+n-2(m+n)}{m+n} = 0$$

$$\Rightarrow \frac{8m+3n-9m+n-2m-2n}{m+n} = 0$$

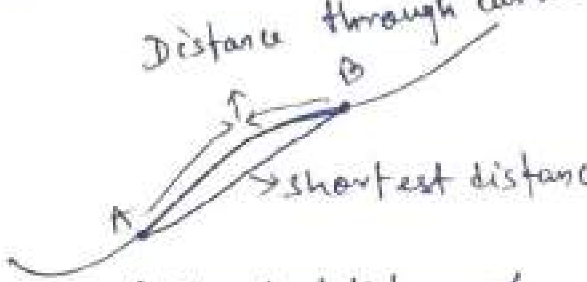
$\Rightarrow -3m + 2n = 0$ $\quad \Rightarrow 3m = 2n$

$\Rightarrow \dfrac{m}{n} = \dfrac{2}{\ldots}$

STRAIGHT LINE

Definition :- In a straight line if we take any two points then the shortest distance between the points is equal to the distance between the pts through the curve.

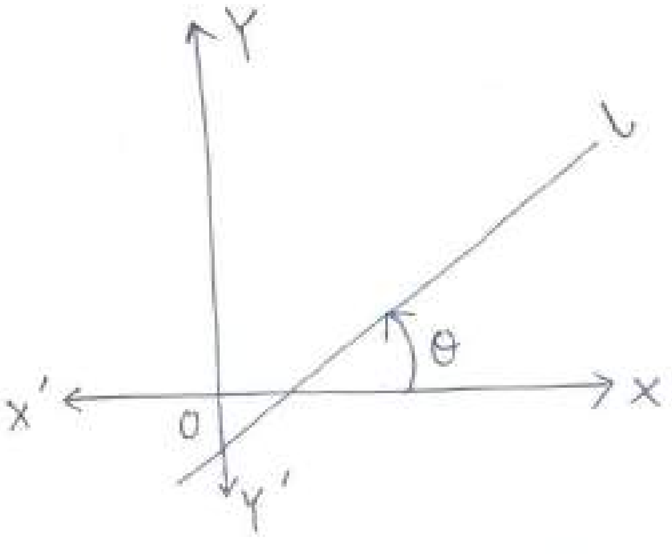
Here shortest distance = Distance through curve (Straight line)



(Shortest distance ≠ distance through curve)

Inclination of a straight line :-

Inclination of a line is the angle made by the line with the positive X-axis and measured in anti clockwise direction.



Here 'θ' is the inclination of the st. line 'L'.

* Inclination of X-axis or any line parallel to X-axis is 0°

* Inclination of Y-axis or line $\parallel^r$ to Y-axis is 90°.

Slope of a line :-

① Slope of a line which makes an angle θ with positive X-axis is $\tan\theta$ and denoted by the letter 'm'.

i.e. if inclination of a line is 'θ'

then its slope is $\boxed{m = \tan\theta}$

* Slope of a line $\parallel^{r}$ to X-axis or X-axis is

$m = \tan 0^\circ = 0$ (or slope of Horizontal lines is 0)

* slope of a line $\parallel^{r}$ to Y-axis or Y-axis is

$m = \tan 90^\circ =$ undefined.

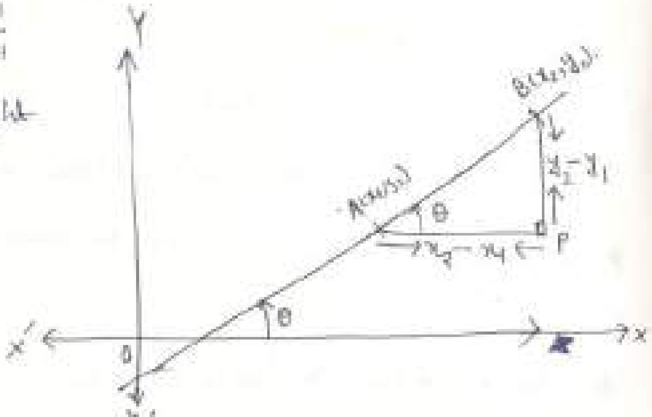
(or slope of Vertical lines is 0)

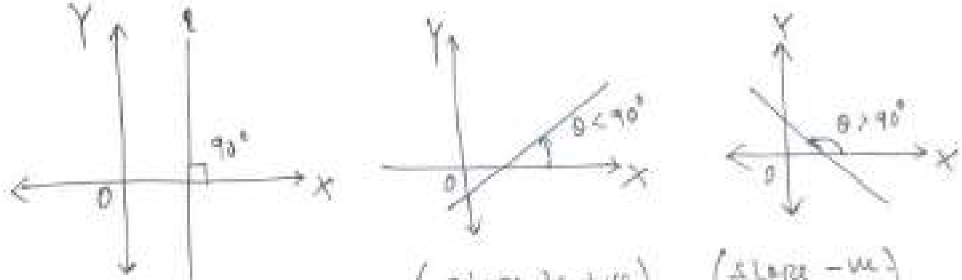
② Slope of line passing through two fixed pts $A(x_1, y_1)$ and $B(x_2, y_2)$

$$\text{slope} = \tan\theta = \frac{P}{B} = \frac{y_2 - y_1}{x_2 - x_1}$$

by considering the right angle Δ ABP

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$



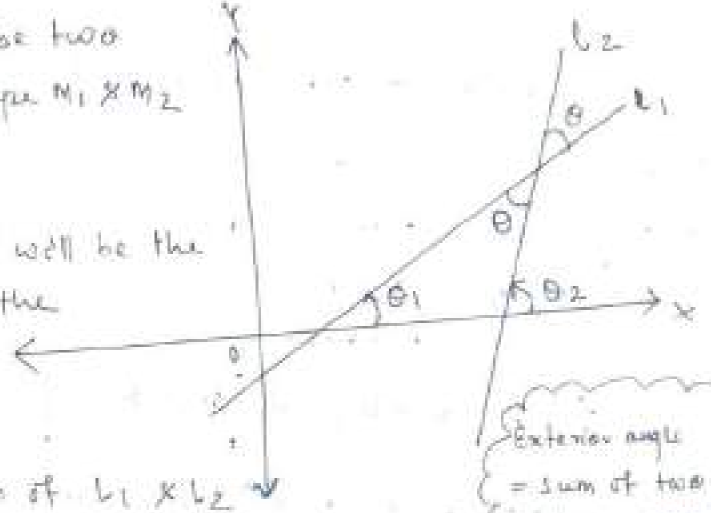


(slope = undefined) (slope is +ve) (slope -ve)

Angle between two lines :-

Let l_1 and l_2 be two lines with slope m_1 & m_2 respectively.

Then Here 'θ' will be the angle between the two lines.



Exterior angle = sum of two interior opposite angles

→ Here inclination of l_1 & l_2 be θ_1 & θ_2 respectively (let).

Then $\theta_2 = \theta + \theta_1$

$\Rightarrow \theta = \theta_2 - \theta_1$

$\Rightarrow \tan\theta = \tan(\theta_2 - \theta_1)$

$$= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \cdot \tan\theta_1}$$

$$\Rightarrow \tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} \quad \left(\begin{array}{l} \text{as } m_1 = \tan\theta_1 \\ m_2 = \tan\theta_2 \end{array} \right)$$

$$\text{or } \tan\theta = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right| \quad (\text{as } \theta \text{ is an acute angle})$$

$$\text{or } \tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\text{or } \boxed{\theta = \tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|}$$

NOTE 1 Condition of parallelism of two lines.

Two lines are parallel

$\Rightarrow$ angle between them is zero.

$$\Rightarrow \tan 0 = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\Rightarrow 0 = m_1 - m_2 \Rightarrow \boxed{m_1 = m_2}$$

NOTE 2 Condition of perpendicularity

Two lines are perpendicular.

$\Rightarrow$ angle between them is 90°

$$\Rightarrow \tan 90^\circ = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \infty = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow 0 = \frac{1 + m_1 m_2}{m_1 - m_2}$$

$$\Rightarrow m_1 m_2 + 1 = 0$$

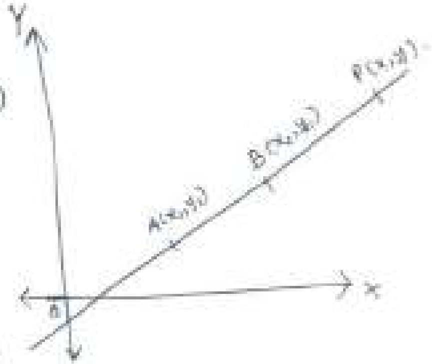
$$\Rightarrow \boxed{m_1 m_2 = -1}$$

EQUATION OF STRAIGHT LINE

① Equation of a line in two pt. form :-

Let L be a line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$

& let's take $P(x, y)$ be a general pt. on the line.



Then slope of AP = slope of AB

$$\Rightarrow \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)} \quad \text{required eq}^n \text{ of the line}$$

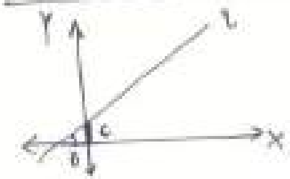
② Equation of line in point-slope form :-

Let m be the slope of the line and the line passing through (x_1, y_1)

then required eq^n is

$$\boxed{y - y_1 = m(x - x_1)}$$

③ We have $y - y_1 = m(x - x_1)$ **Slope-intercept form**



$\Rightarrow y - y_1 = mx - mx_1$

$\Rightarrow y = mx - mx_1 + y_1$

$\Rightarrow \boxed{y = mx + c}$ also the eqⁿ of straight line

where coefficient of x is slope of line l.

$\Rightarrow$ So eqⁿ of straight line is a linear eqⁿ in x & y.

i.e. $mx - y + c = 0$

④ **General Eqⁿ of straight line.**

Any linear eqⁿ in x and y is the eqⁿ of a straight line.

i.e. $\boxed{ax + by + c = 0}$ is the required eqⁿ.

NOTE 1:- $by = -ax - c$

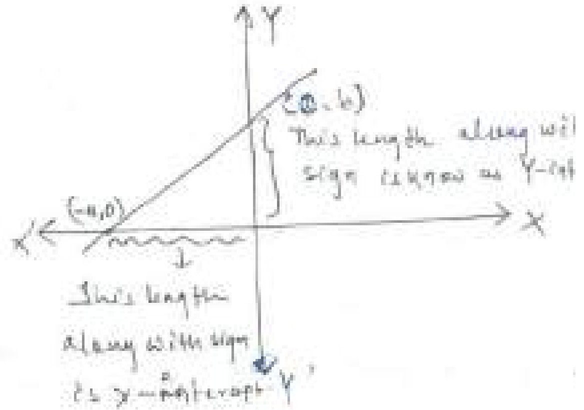
$y = \left(-\frac{a}{b}\right)x + \left(-c/b\right)$

which is in slope-intercept form.

Then $\boxed{\text{slope} = -a/b}$

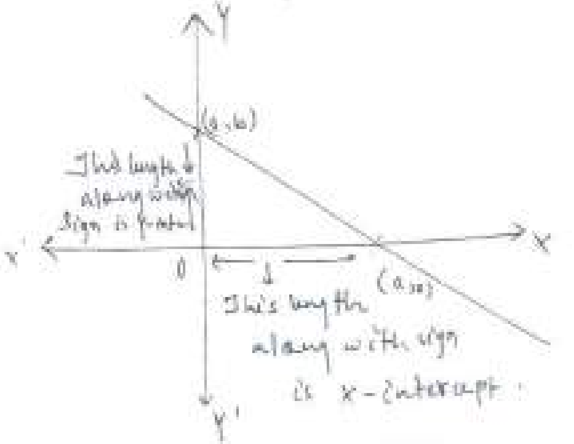
$\boxed{y\text{-intercept} = -c/b}$

⑤ **Intercept form**



x-intercept = $-a$

Y-intercept = b

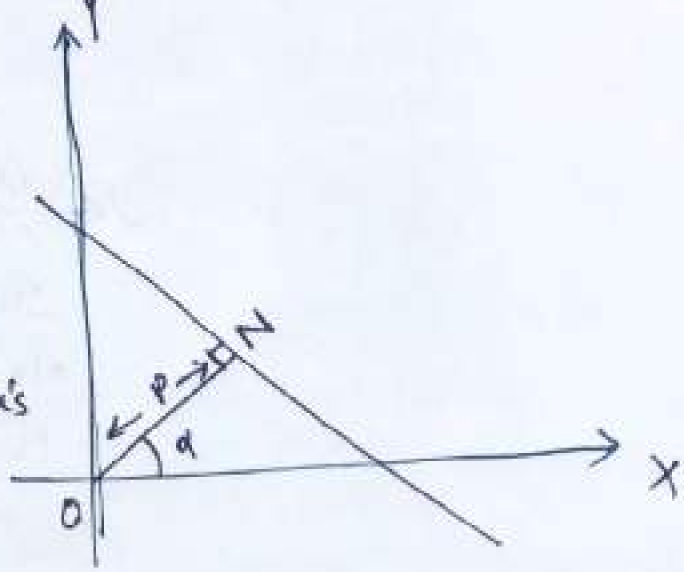


The eqⁿ of the line $\boxed{\frac{x}{a} + \frac{y}{b} = 1}$

(using two point form)

⑥ Normal / perpendicular form :-

Let p be the length of perpendicular from the origin to a given line and α be the angle made by this perpendicular line with the x-axis



then eqn of the line $x \cos\alpha + y \sin\alpha = p$

Reduction of General form to standard form

① Reduction of General form to slope intercept form.

General form $\Rightarrow ax+by+c=0$

$$\Rightarrow by = -ax - c$$

$$\Rightarrow y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right)$$

so here $\boxed{\begin{array}{l} \text{slope} = -a/b \\ y\text{-intercept} = -c/b \end{array}}$

② Intercept form

General form is

$$ax+by+c=0$$

$$\Rightarrow ax+by=-c$$

$$\Rightarrow \frac{ax}{-c} + \frac{by}{-c} = 1$$

$$\Rightarrow \frac{x}{(-c/a)} + \frac{y}{(-c/b)} = 1$$

So here $\boxed{\begin{array}{l} x\text{-intercept} = -c/a \\ Y\text{-intercept} = -c/b \end{array}}$

* Another condition of parallelism of two line in general form

Two lines are $\parallel^r$

then $m_1 = m_2$

$$\Rightarrow \frac{-a_1}{b_1} = \frac{-a_2}{b_2}$$

$$\Rightarrow \boxed{\frac{a_1}{b_1} = \frac{a_2}{b_2}}$$

* Another condition of perpendicularity

Let $a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$ are two perpendicular

lines then $m_1 m_2 = -1$

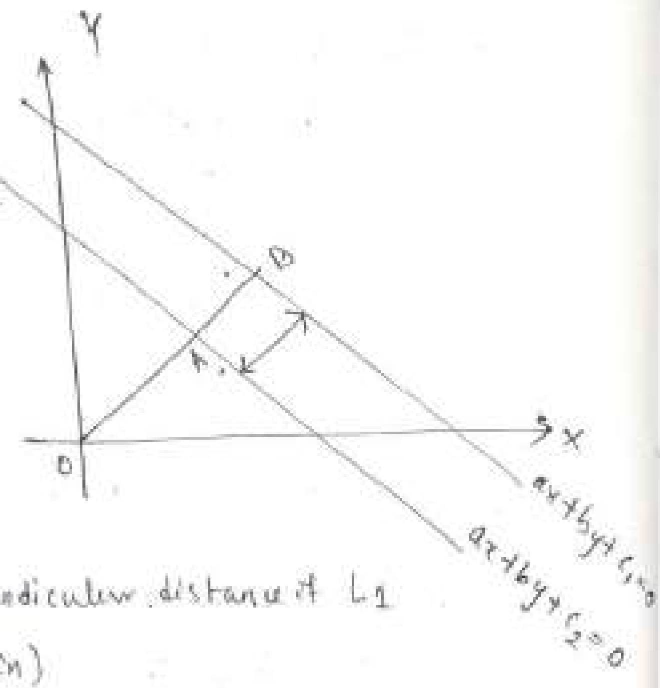
$$\Rightarrow \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right) = -1$$

$$\Rightarrow a_1 a_2 = -b_1 b_2$$

$$\Rightarrow \boxed{a_1 a_2 + b_1 b_2 = 0}$$

Distance between two parallel lines :-

Distance = shortest/ perpendicular distance between two line.



Here OB (Perpendicular distance of L_1 from origin)

$$OB = \left|\frac{c_1}{\sqrt{a^2+b^2}}\right|$$

OA = Perpendicular distance of L_2 from origin.

$$OA = \left|\frac{c_2}{\sqrt{a^2+b^2}}\right|$$

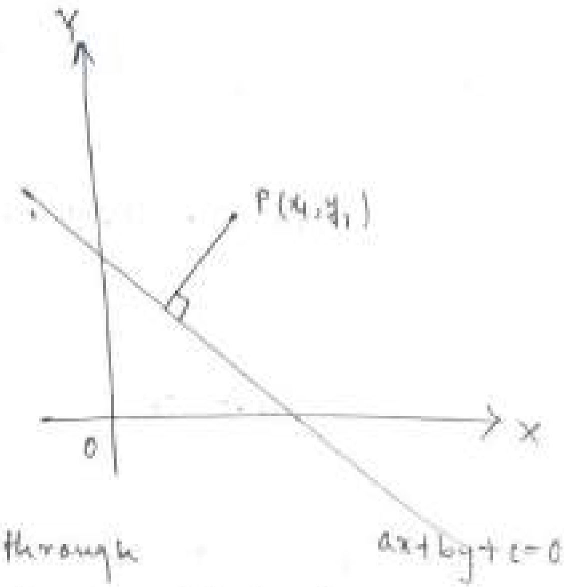
$$AB = OB - OA = \left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|$$

So $\boxed{D = \frac{c_1-c_2}{\sqrt{a^2+b^2}}}$

(When using this formula coefficient of x and y must be same in both lines.)

Distance between a pt. (x_1, y_1) and a line $ax+by+c=0$

(Perpendicular Distance)



Eqⁿ of the line passing through pt $P(x_1, y_1)$ and parallel to $ax+by+c=0$

Then slope will be same i.e $-a/b$

Then using slope-intercept form.

eqⁿ of the required line.

$$y - y_1 = -\frac{a}{b}(x - x_1)$$

$$\Rightarrow ax + by - ax_1 - by_1 = 0$$

$$\Rightarrow ax + by - (ax_1 + by_1) = 0$$

Then Distance between the lines.

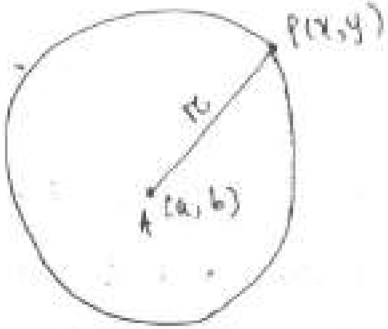
$$D = \left|\frac{c - \{-(ax_1+by_1)\}}{\sqrt{a^2+b^2}}\right| = \boxed{\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|}$$

CIRCLE

Definition:- circle is a locus of a point 'P' which moves in such a way that it's distance from a fixed point is always constant.

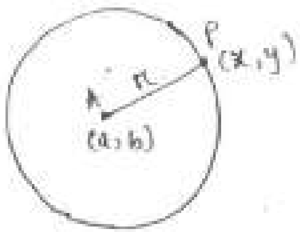
where fixed pt - (a, b) is centre.

r = radius = Distance AP



Standard Equation of a circle:-

Eq^n of the circle whose centre (a, b) and radius is 'r'



Distance of AP

$$r = \sqrt{(x-a)^2 + (y-b)^2}$$

Squaring both sides

$$\Rightarrow \boxed{r^2 = (x-a)^2 + (y-b)^2}$$

which is the required eq^n of the circle.

General Eq^n of the circle:-

Standard form.

$$(x-a)^2 + (y-b)^2 = r^2$$

$$\Rightarrow x^2 + 2(-a)x + a^2 + y^2 - 2by + b^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

Let $-a = g$

$-b = f$

$a^2 + b^2 - r^2 = c$

$$\Rightarrow \boxed{x^2 + y^2 + 2gx + 2fy + c = 0}$$

required eq^n of circle.

Then centre = (a, b) = $\boxed{(-g, -f) = \text{centre}}$

radius = r

We have $a^2 + b^2 - r^2 = c$

$$\Rightarrow r^2 = a^2 + b^2 - c$$

$$= (-g)^2 + (-f)^2 - c$$

$$\Rightarrow \boxed{r = \sqrt{g^2 + f^2 - c}}$$

NOTE :- $r = \sqrt{g^2+f^2-c}$

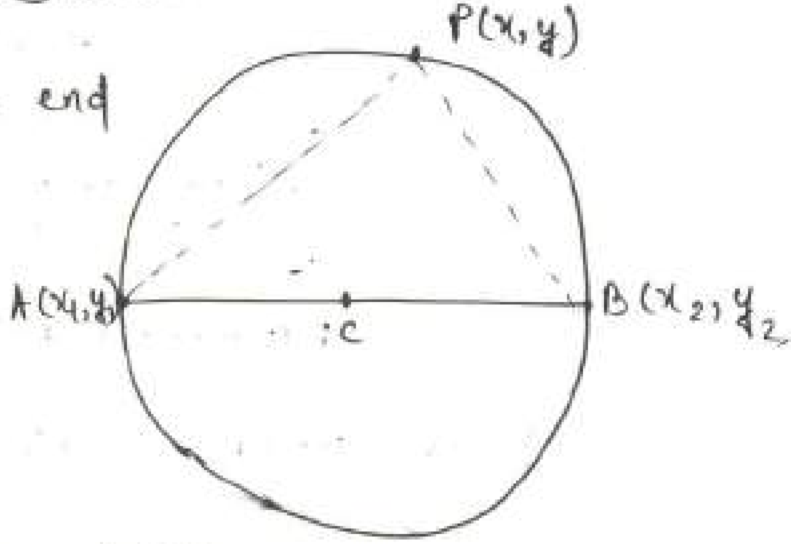
(i) if $g^2+f^2-c > 0 \Rightarrow$ real circle.

(ii) if $g^2+f^2-c < 0 \Rightarrow$ imaginary circle.

(iii) $g^2+f^2-c = 0 \Rightarrow$ Point circle. of radius '0'.

Diametrical Eq^n of a circle :-

Eq^n of a circle whose end pts of diametre are (x_1, y_1) and (x_2, y_2)



Here $AP \perp BP$

$\Rightarrow$ slope of AP · slope of BP = −1

$$\Rightarrow \frac{y-y_1}{x-x_1} \cdot \frac{y-y_2}{x-x_2} = -1$$

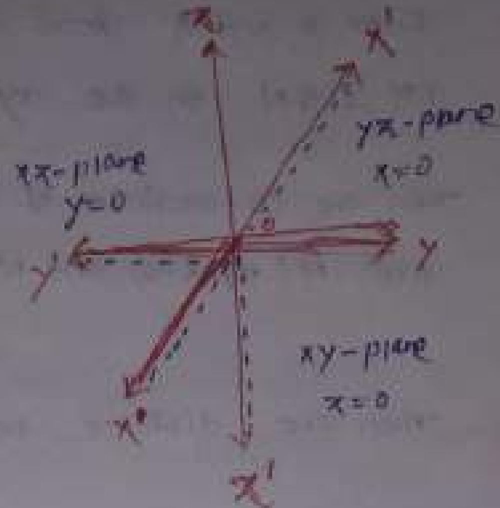
$$\Rightarrow (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

$$\Rightarrow \boxed{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0}$$

Co-ordinate Geometry in three Dimensions:

We take three perpendicular lines as axes.

O, the point of intersection is called origin.



$x'Ox$ is called x-axis.

$y'Oy$ is called y-axis.

$z'Oz$ is called z-axis.

The three lines taken together are called rectangular co-ordinate axes.

x-axis written as $(x, 0, 0)$.

y-axis written as $(y, 0, 0)$.

z-axis written as $(z, 0, 0)$.

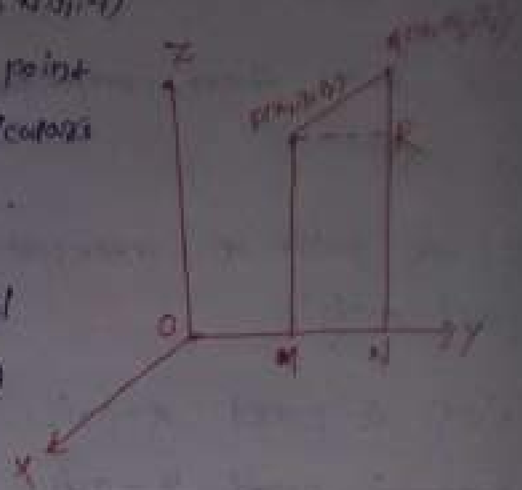
Distance Formula, Division Formula:

Theorem-1 (Distance Formula)

Prove that the distance bet^n the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by.

$$\boxed{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Proof: Let O be origin and Let $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ be the given point. From P and Q draw perpendiculars PM & QN on the xy-plane.



Then the co-ordinate of M and N are $M(x_1,y_1,0)$ and $N(x_2,y_2,0)$

Then the distance bet^n $MN = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(0-0)^2}$

$$\Rightarrow MN^2 = (x_2-x_1)^2+(y_2-y_1)^2$$

Now, from P draw $PR \perp QN$.

Then PR is parallel and equal to MN.

Now, in Right angle triangle PRQ, we have,

$$PQ^2 = PR^2 + RQ^2$$
$$= MN^2 + (QN - RN)^2$$
$$= MN^2 + (QN - PM)^2$$

$$\boxed{PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}} \quad [\because PM = z_1 \ \& \ QN = z_2]$$

Corollary:- The distance of the point $P(x,y,z)$ ~~and~~ from the origin $O(0,0,0)$ is

$$OP = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$
$$= \sqrt{x^2+y^2+z^2}$$

Ex-1 Find the distance betⁿ two points (2,3,5) and (4,3,1)

Solⁿ Distance betⁿ two points (2,3,5) and (4,3,1) is

$= \sqrt{(4-2)^2 + (3-3)^2 + (1-5)^2}$

$= \sqrt{2^2 + 0^2 + (-4)^2}$

$= \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$ Ans

Ex-2 Find the value of 'x' if distances betⁿ two points $(x,-8,4)$ and $(3,-5,4)$ is 5

Solⁿ Given points, $(x,-8,4)$ and $(3,-5,4)$

The distance betⁿ these points = 5

$\sqrt{(x-3)^2 + (-8+5)^2 + (4-4)^2} = 5$

or, $(x-3)^2 + 9 + 0 = 25 \Rightarrow (x-3)^2 = 16$

or, $(x-3) = \pm 4$ or, $x = 7, -1$.

Ex-3 Show that the points $A(-2,-6,-7)$, $B(4,-4,-5)$, $C(7,-3,-4)$ are collinear

Solⁿ We have

$|AB| = \sqrt{(4+2)^2 + (-4+6)^2 + (-5+7)^2}$

$= \sqrt{6^2 + 2^2 + 2^2}$

$= \sqrt{36+4+4}$

$= \sqrt{44} = 2\sqrt{11}$

$|BC| = \sqrt{(7-4)^2 + (-3+4)^2 + (-4+5)^2}$

$= \sqrt{3^2 + 1^2 + 1^2}$

$= \sqrt{9+1+1}$

$= \sqrt{11}$

$$|CA| = \sqrt{(2-7)^2 + (-6+3)^2 + (-7+4)^2}$$

$$= \sqrt{81+9+9} = \sqrt{99} = 3\sqrt{11}$$

$$\therefore \quad |AB| + |BC| = |CA|$$

Hence the points A, B, C are collinear.

Ex-4 imp Show that the points $A(1,2,3)$, $B(-1,-2,-1)$, $C(2,3,2)$ and $D(4,7,6)$ are the vertices of a parallelogram ABCD, but it is not a rectangle.

Soln We have,

$$AB = \sqrt{(-1-1)^2 + (-2-2)^2 + (-1-3)^2} = \sqrt{4+16+16} = 6$$

$$BC = \sqrt{(2+1)^2 + (3+2)^2 + (2+1)^2} = \sqrt{9+25+9} = \sqrt{43}$$

$$CD = \sqrt{(4-2)^2 + (7-3)^2 + (6-2)^2} = \sqrt{4+16+16} = 6$$

$$DA = \sqrt{(1-4)^2 + (2-7)^2 + (3-6)^2} = \sqrt{9+25+9} = \sqrt{43}$$

$\therefore$ $AB = CD$ and $BC = DA$. Hence opposite sides are equal.

$\therefore$ ABCD is a parallelogram.

Again $$AC = \sqrt{(2-1)^2 + (3-2)^2 + (2-3)^2}$$

$$= \sqrt{1+1+1} = \sqrt{3}$$

$$BD = \sqrt{(4+1)^2 + (7+2)^2 + (6+1)^2}$$

$$= \sqrt{25+81+49} = \sqrt{155}$$

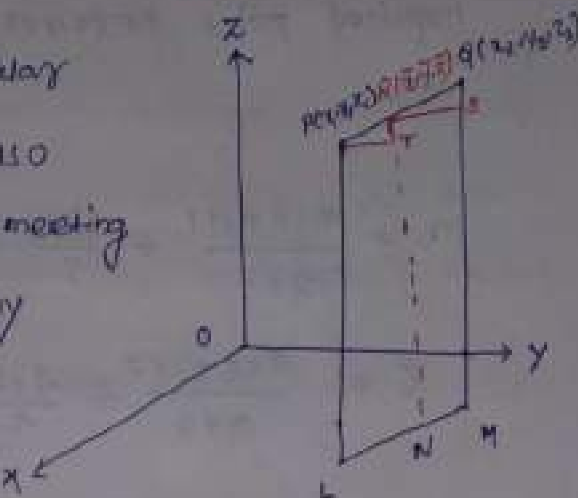
$\therefore$ $AC \neq BD$ i.e the diagonal are not equal;

Division Formula :- (Ratio formula)

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let $R(\bar{x}, \bar{y}, \bar{z})$ be a point on PQ dividing it in the ratio $m:n$ prove that!

$$\bar{x} = \frac{mx_2 + nx_1}{m+n}, \quad \bar{y} = \frac{my_2 + ny_1}{m+n}, \quad \bar{z} = \frac{mz_2 + nz_1}{m+n}$$

Lets draw P, Q & R perpendicular PL, QM and RN on xy-plane. Also draw RS $\perp$ QM and PT $\perp$ RN meeting QM & RN at S and T respectively



PRT and RQS are similar Triangles.

We have $\frac{RT}{QS} = \frac{PR}{RQ}$

$$\Rightarrow \frac{RN - TN}{QM - SM} = \frac{m}{n}$$

$$\Rightarrow \frac{\bar{z} - z_1}{z_2 - \bar{z}} = \frac{m}{n}$$

$$\Rightarrow n\bar{z} - nz_1 = mz_2 - m\bar{z}$$

$$\Rightarrow m\bar{z} + n\bar{z} = mz_2 + nz_2$$

$$\Rightarrow \bar{z}(m+n) = z_2(m+n)$$

$$\Rightarrow \bar{z} = \frac{mz_2 + nz_1}{\bar{z}}$$

Similarly $\bar{x} = \frac{mx_2 + nx_1}{m+n}$, $\bar{y} = \frac{my_2 + ny_1}{m+n}$

For internal division - Replacing n by $-n$.

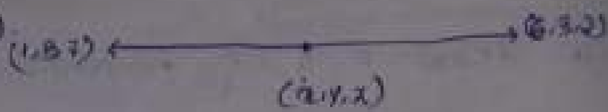
$$\bar{x} = \frac{mx_2 - nx_1}{m-n}$$

$$\bar{y} = \frac{my_2 - ny_1}{m-n}$$

$$\bar{z} = \frac{mz_2 - nz_1}{m-n}.$$

Ex-1 Find the co-ordinates of a point which divides the points $(1,3,7)$, $(6,3,2)$ in the ratio $2:3$.

Sol^n Let the co-ordinate of the required point be (x,y,z)



$$x = \frac{2\times 6 + 3\times 1}{2+3} = \frac{12+3}{5} = 3, \quad y = \frac{2\times 3 + 3\times 3}{2+3} = \frac{6+9}{5} = 3$$

$$z = \frac{2\times 2 + 3\times 7}{2+3} = \frac{4+21}{5} = 5$$

$\therefore$ Required point is $(3,3,5)$.

Ex-2 Find the ratio in which the line joining the points $(4,4,-10)$ and $(-2,2,4)$ is divided by

(a) the yz-plane (b) $x+y+z=3$.

Sol^n The given points are $(4,4,-10)$ and $(-2,2,4)$.

Let the ratio is $k:1$.

Then
$$x = \frac{-2k+4}{k+1}$$
$$y = \frac{2k+4}{k+1}$$
$$z = \frac{4k-10}{k+1}$$

(a) If the plane lies in yz-plane then $x=0$

$\therefore \frac{-2k+4}{k+1} = 0$ or, $-2k+4=0$

$\Rightarrow -2k = -4$

$\Rightarrow \boxed{k=2}$

$\therefore$ The required Ratio is 2:1.

(b) $x+y+z = 3$.

$$\Rightarrow \frac{-2k+4}{k+1} + \frac{2k+4}{k+1} + \frac{4k-10}{k+1} = 3$$

$$\Rightarrow \frac{-2k+4+2k+4+4k-10}{k+1} = 3$$

$$\Rightarrow \frac{4k-2}{k+1} = 3$$

$\Rightarrow 4k-2 = 3k+3$

$\Rightarrow 4k-3k = 3+2$

$\Rightarrow \boxed{k=5}$.

$\therefore$ The required ratio is 5:1.

Ex-3 Find the ratio in which the line through (2,4,5) (3,5,-4) is divided by xy-plane

H.W.

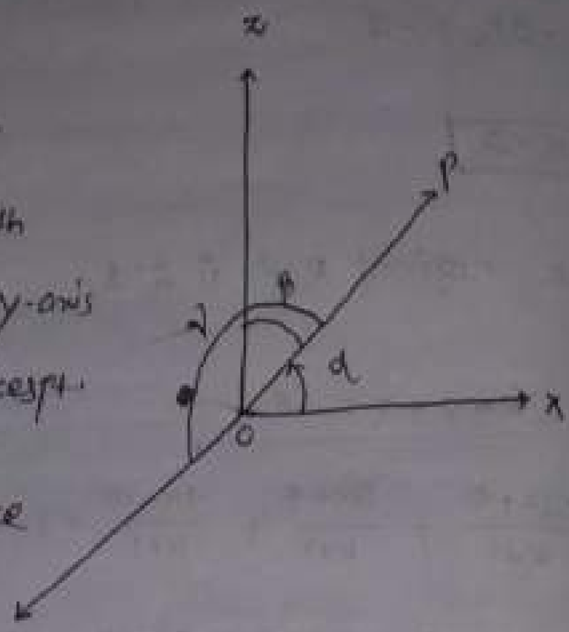
Ex-4 Find the ratio in which the line Joining the points (2,-3,1), (3,-4,-5) is divided by the plane $2x+y+z=7$.

H.W.

Ex-5 Find the ratio in which the line segment Joining the points (4,3,2) & (1,2,-3) is divided by xy-plane.

H.W.

Direction cosine:



Let $\overline{OP}$ be a st. line.

$\overline{OP}$ makes angle α with OX-axis and β with oy-axis and γ with oz-axis respt.

Then $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called the direction cosines (d.c's) of the line.

The (d.c's) of the line are denoted by l, m, n.

$\therefore l = \cos\alpha, m = \cos\beta, n = \cos\gamma$.

In particular the d.c's of x-axis are 1, 0, 0.

similarly the d.c's of y-axis and z-axis are 0, 1, 0 and 0, 0, 1 respectively.

Note: $\boxed{l^2 + m^2 + n^2 = 1}$

Direction Ratios: The number a, b, c are called d.r's. They are ~~the~~ written in the form ⟨a, b, c⟩ or [a, b, c] or a, b, c.

$$\therefore \frac{a}{l} = \frac{b}{m} = \frac{c}{n} = K \text{ (say)}$$

$$\Rightarrow a = Kl, \ b = Km, \ c = Kn \quad \text{——— (1)}$$

where 'k' is the constant of proportionality. Squaring & adding these eq^n's we get

$$a^2+b^2+c^2 = k^2(l^2+m^2+n^2)$$

$$\therefore a^2+b^2+c^2 = k^2 \quad [\because l^2+m^2+n^2=1]$$

$$\Rightarrow k = \pm\sqrt{a^2+b^2+c^2}$$

$$l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}$$

Ex If a line has direction ratio $\langle -18, 12, -4 \rangle$. then determine its d.c's.

Direction cosines & Direction ratio of the line segment Joining 2 pts.

Suppose $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ are two points

The direction ratio's are $\langle x_2-x_1, y_2-y_1, z_2-z_1 \rangle$

Angle betn two lines :-

① The Angle betn two lines having d.c's l_1, m_1, n_1 & l_2, m_2, n_2
$\theta \to$ Angle

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$\Rightarrow \boxed{\theta = \cos^{-1}(l_1 l_2 + m_1 m_2 + n_1 n_2)}$$

② If $\langle a_1, b_1, c_1 \rangle$ and $\langle a_2, b_2, c_2 \rangle$ be d.r's of two lines then the d.c's are

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2+b_1^2+c_1^2} \times \sqrt{a_2^2+b_2^2+c_2^2}}\right)}$$

Ex-15 Find the acute angle betn the lines whose d.r.'s are $(1,1,2)$ and $(\sqrt{3}-1, -\sqrt{3}-1, 4)$ resp.

$(\sqrt{3}+1)^2 = 3+1+2\sqrt{3}$

Soln

Here $\langle a_1, b_1, c_1 \rangle = \langle 1, 1, 2 \rangle$ and

$\langle a_2, b_2, c_2 \rangle = \langle \sqrt{3}-1, -\sqrt{3}-1, 4 \rangle$

Then $\cos\theta = \dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2} \times \sqrt{a_2^2+b_2^2+c_2^2}} = \dfrac{\sqrt{3}-1+(-\sqrt{3}-1)+8}{\sqrt{6} \times \sqrt{(\sqrt{3}-1)^2+(-\sqrt{3}-1)^2+4^2}}$

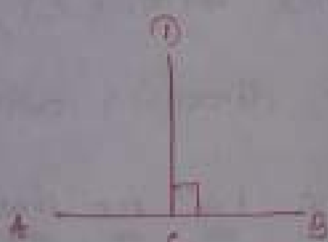
$= \dfrac{6}{\sqrt{6} \times \sqrt{3+1-2\sqrt{3}+3+1+2\sqrt{3}+16}} = \dfrac{6}{\sqrt{6} \times \sqrt{24}}$

$= \dfrac{6}{\sqrt{144}} = \dfrac{6}{12} = \dfrac{1}{2}$

$\therefore \cos\theta = 1/2 \Rightarrow \theta = \cos^{-1}\frac{1}{2} = 60^\circ$

$\therefore$ Hence the reqd. Angle $\theta = 60^\circ$

Case-1 Condition for $\perp^r$:



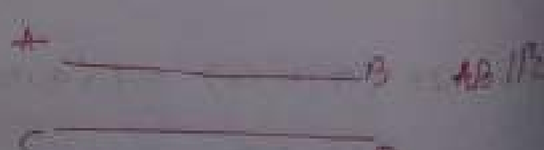
Here $\theta = 90^\circ$

$\cos\theta = \cos 90^\circ = 0$

$\boxed{l_1 l_2 + m_1 m_2 + n_1 n_2 = 0}$

In term's of d.r's of the lines the $\perp^r$ condn is

$\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$



Case-2 Condn for $\parallel^r$

Here $\theta = 0^\circ$

$\cos 0^\circ = 1$.

$\boxed{\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2} = 1}$

For d.r's

$\boxed{\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = 1}$

Unit-4 Plane

Plane- A plane is defined as surface such that the st. line joining any two points on the surface lies on it.

Note $x=0$, is the eqⁿ of yz-plane
$y=0$, is the eqⁿ of zx-plane
$z=0$, is the eqⁿ of xy-plane.

Normal form: The eqⁿ of plane is

$$\boxed{lx+my+nz=p} \quad \{\because l^2+m^2+n^2=1\}$$

Thm-3 Equation of the plane ~~through~~ making off intercepts a, b, c on the co-ordinate axes is

$$\boxed{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1}.$$

Plane through a given point:

The eq of any plane through (x_1, y_1, z_1) is

$$\boxed{a(x-x_1)+b(y-y_1)+c(z-z_1)=0}.$$

Thm-4 Find the eq of the plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

This is the reqd. eq of the plane

Thm-5: To reduce the eq^n $ax+by+cz+d=0$ of the plane from general form to normal form.

Sol^n: The given eq^n is

$$ax+by+cz+d=0$$

Hence, in general $\langle a,b,c \rangle$ are the drcs of the normal to the plane. Let $\langle l,m,n \rangle$ be the direction cosine of the normal so, we have

$$l = \frac{a}{\pm\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\pm\sqrt{a^2+b^2+c^2}}$$

$$n = \frac{c}{\pm\sqrt{a^2+b^2+c^2}}.$$

Plane passing through the intersection of two points:-

Consider two intersecting planes given by the eq^s

$$a_1x+b_1y+c_1z+d_1=0 \quad \text{———} (1)$$

$$a_2x+b_2y+c_2z+d_2=0 \quad \text{———} (2)$$

We consider the eq^n

$$(a_1x+b_1y+c_1z+d_1)+k(a_2x+b_2y+c_2z+d_2)=0$$

where k is a parameter

i.e. $(a_1+ka_2)x+(b_1+kb_2)y+(c_1+kc_2)z+(d_1+kd_2)=0$

which is also a plane.

Angle between two planes :

– Angle between two planes is equal to the angle between their normals.

If the eq^n of two planes be

$a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$

If 'θ' be the angle betn the planes

$$\boxed{\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}}$$

(i) The plane will be parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

(ii) and perpendicular if $\cos\theta = 0$

i.e $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(iii) The planes are identical if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$$

Note-1 : The distance of the point (x_0, y_0, z_0) from the plane $ax + by + cz + d = 0$ is given by

$$D = \left|\frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2+b^2+c^2}}\right|$$

Note-2 : Distance betn two parallel planes $ax+by+cz+d=0$ and $ax+by+cz+d_1=0$ is given by

$$D = \left|\frac{d - d_1}{\sqrt{a^2+b^2+c^2}}\right|$$

Examples

Ex-1 Find the eqⁿ of the plane which passes through $(4,-2,1)$ and is perpendicular to the line whose direction ratios are $7, 2, -3$.

Solⁿ Equation of the plane through $(4,-2,1)$ is

$$a(x-4)+b(y+2)+c(z-1)=0 \quad \text{——(i)}$$

Since the drs of the normal to the plane are $7, 2, -3$.

$\therefore$ $a=7,\ b=2,\ c=-3$.

putting these values of a, b and c (i) we get

$$7(x-4)+2(y+2)-3(z-1)=0$$

or, $7x+2y-3z-21=0$.

is the reqd. eqⁿ of the plane.

Imp Ex-2 Find the eqⁿ of the plane which passes through the point $(1,-1,4)$ and is parallel to the plane $2x-3y+7z=11$.

Solⁿ Any parallel to the plane $2x-3y+7z-11=0$ is of the form $2x-3y+7z+k=0$ ——(1)

Since it passes through $(1,-1,4)$

$\therefore$ $2(1)-3(-1)+7(4)+k=0$

or $2+3+28+k=0$

$\Rightarrow k=-33$

put $k=-33$ in eqⁿ ① we get

$$2x-3y+7z-33=0$$

Ex-3 Find the eqⁿ of the plane containing the line of intersection of the planes $x+y+z+1=0$, $2x-3y+5z-2=0$ and passing through the point $(-1,2,1)$.

Solⁿ Eqⁿ of any plane passing through the line of intersection of the planes

$$x+y+z+1=0 \quad \text{and} \quad 2x-3y+5z-2=0$$

and passing through $(-1,2,1)$

$$\Rightarrow (x+y+z+1)+\lambda(2x-3y+5z-2)=0 \quad \text{——— (i)}$$

since (i) also passes through $(-1,2,1)$

$$\therefore (-1+2+1+1)+\lambda(-2-6+5-2)=0$$

or $3+\lambda(-3)=0$ or $\boxed{\lambda=\frac{3}{5}}$

putting this values of $\frac{3}{5}$ in (i) we get

$$(x+y+z+1)+\frac{3}{5}(2x-3y+5z-2)=0$$

or $5x+5y+5z+5+6x-9y+15z-6=0$

or $11x-4y+20z-1=0$ is the reqd. eqⁿ of the plane.

Imp

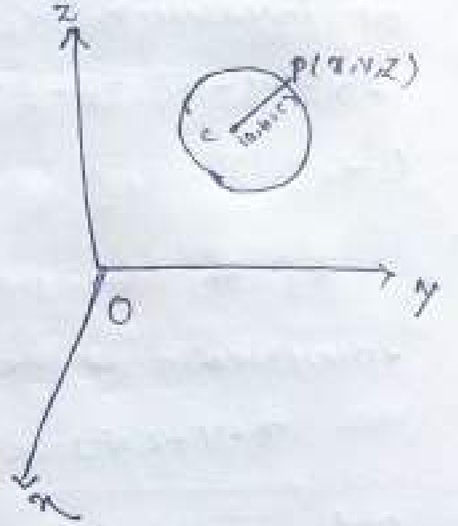
Ex-4 Find the eqⁿ of the plane passing through the point $(-1,-1,2)$ and ⊥ʳ to the planes

$$3x+2y-3z=1$$

and $5x-4y+z=5$

Sphere

A sphere is the locus of a point in space which moves in such a way that it remains always at a constant distance from a fixed point.



The fixed point is called the centre and the constant distance is called radius of the sphere.

Them-1 The eq^n of the sphere with centre at the point (a,b,c) and radius r is given by

$$\boxed{(x-a)^2+(y-b)^2+(z-c)^2=r^2}$$

General eq^n of a sphere:-

General eq^n of a sphere expressed in the form

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

Hence the centre of the sphere is $(-u,-v,-w)$ and radius of the sphere is $\sqrt{u^2+v^2+w^2-d}$.

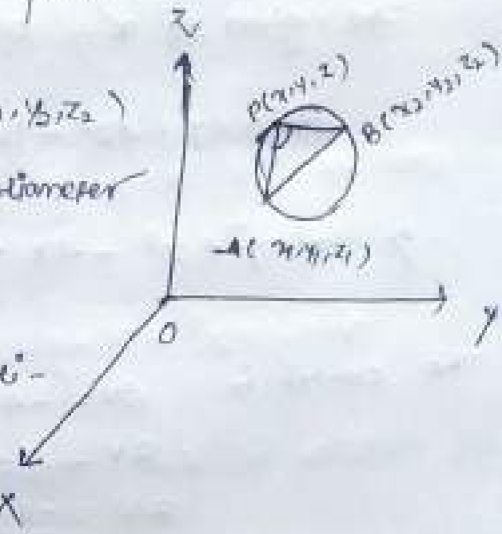
Note

If $u^2+v^2+w^2<d$ sphere may be called as imaginary sphere.

If $u^2+v^2+w^2>d$ sphere may be called as real sphere.

Thm-2 The co-ordinates of end point of a diameter of a sphere are (x_1, y_1, z_1) and (x_2, y_2, z_2). find the eq^n of the sphere.

Sol^n Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two end points of diameter of a sphere.



using condition of perpendi-cularity

$$\boxed{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0}$$

Thm-3 To find the eq^n of the sphere through four given points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4).

Sol^n Let the eq^n of the sphere be
$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$
since it passes through (x_1,y_1,z_1) (x_2,y_2,z_2), (x_3,y_3,z_3) and (x_4,y_4,z_4) is given by

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0 \quad \text{——(1)}$$
$$x_2^2+y_2^2+z_2^2+2ux_2+2vy_2+2wz_2+d=0 \quad \text{——(2)}$$
$$x_3^2+y_3^2+z_3^2+2ux_3+2vy_3+2wz_3+d=0 \quad \text{——(3)}$$
$$x_4^2+y_4^2+z_4^2+2ux_4+2vy_4+2wz_4+d=0 \quad \text{——(4)}$$

Solving these eq^ns we get the reqd. sphere

Ex-1 Find the centre and radius of the sphere

$$4x^2+4y^2+4z^2-16x-24z+3=0$$

Sol^n The given eq^n of sphere is

$$4x^2+4y^2+4z^2-16x-24z+3=0$$

$$\Rightarrow x^2+y^2+z^2-4x-6z+3/4=0$$

The centre is $2u=-4 \Rightarrow u=-2$

$2v=0 \Rightarrow v=0$

$2w=-6 \Rightarrow w=-3$

$$(-u,-v,-w)=(2,0,3)$$

Radius is $=\sqrt{u^2+v^2+w^2-d}=\sqrt{4+0+9-\frac{3}{4}}$

$$=\sqrt{13-\frac{3}{4}}=\sqrt{\frac{49}{4}}=7/2.$$

Hence the centre is $(2,0,3)$ and radius is $7/2$.

Ex-2 Find the eq^n of the sphere on the Joint of $(2,3,5)$ and $(4,9,-3)$ as diameter?

Sol^n We know that eq^n of a sphere whose end points of the diameter are (x_1,y_1,z_1) and (x_2,y_2,z_2) is.

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

$\therefore$ Eqn of the sphere whose end points of the diameter are (2,3,5) and (4,9,-3) is

$$(x-2)(x-4)+(y-3)(y-9)+(z-5)(z+3)=0$$

$$\Rightarrow x^2+y^2+z^2-6x-12y-2z+8+27-15=0$$

$$\Rightarrow x^2+y^2+z^2-6x-12y-2z+20=0$$

is the reqd. eqn of the sphere.

Ex-3 Find the eqn of the sphere which passes through the points (0,0,0), (0,1,0), (1,0,0) & (0,0,1).

Soln Let the eqn of sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

Since (0,0,0) lies on (i)

$\therefore$ $d=0$ ——— (2)

Again (0,1,0) lies on (i)

$\therefore$ $1+2v+d=0$

or $1+2v=0 \Rightarrow v=-\frac{1}{2}$ $(\because d=0)$

Also (1,0,0) lies on (i)

$\therefore$ $1+2u+d=0$

or $1+2u=0$

$\Rightarrow u=-\frac{1}{2}$.

Again $(0,0,1)$ lies on (i)

$\therefore \quad 1 + 2w + d = 0$

$\Rightarrow 1 + 2w = 0 \Rightarrow w = -\frac{1}{2} \quad (\because d = 0)$

put $u = v = w = -\frac{1}{2}$ and $d = 0$ in (i) we get

$$x^2 + y^2 + z^2 - x - y - z = 0$$

which is the reqd. eqn of the sphere.

Ex-4 Find the eqn of the sphere with centre $(3,-2,5)$ and radius 4.

Soln The reqd. eqn of the sphere is

$$(x-3)^2 + (y+2)^2 + (z-5)^2 = 4^2 = 16$$

$$\Rightarrow x^2 + y^2 + z^2 - 6x + 4y - 10z + 22 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - 6x + 4y - 10z + 22 = 0$$

Imp

Ex-5 Find the eqn of the sphere which passes through the points $(0,0,0)$, $(-a,b,c)$, $(a,-b,c)$ and $(a,b,-c)$.