LECTURES NOTE

SUB: ENGINEERING MATHEMATICS II

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ENGINEERING MATHEMATICS

LECTURE NOTE

Based on New syllabus (2018-19) circulated by SCTE&VT, Odisha for 1st and 2nd Semester Diploma Engineering courses approved by AICTE, New Delhi

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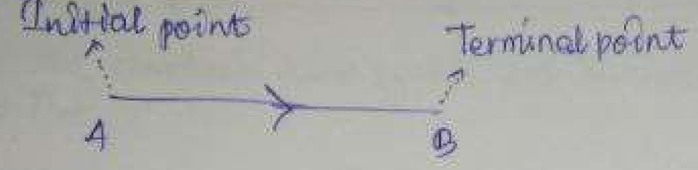
Vector Algebra

Vector :- Physical quantities having both magnitude and direction are called vectors.
Ex- Weight, displacement, velocity, acceleration etc.

Scalar :- Physical quantities having only magnitude and no direction are called scalars.

Ex- Speed, mass, volume, energy etc.

Representation of a vector



Generally a vector is denoted by a small letter with arrow head.

$\overrightarrow{AB} = \vec{a}$

Here, magnitude of $\overrightarrow{AB} = \overline{AB}$
Direction is from A towards B.

Types of vectors

(i) Zero vector / Null vector :- A vector ~~which has~~ whose magnitude is '0' is called zero vector.

If $\vec{u}$ is a zero vector than $\underbrace{|\vec{u}|}_{\text{Magnitude of } \vec{u}} = 0$

* A zero vector has no definite direction

(ii) Unit vector :- A vector whose magnitude is unity or '1' is called a unit vector.

* If $\vec{u}$ is a unit vector then $|\vec{u}| = 1$

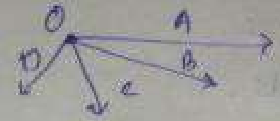
* Unit vector of $\vec{u}$ is denoted by $\hat{u}$ and is defined by.

$$\boxed{\hat{u} = \frac{\vec{u}}{|\vec{u}|}} \quad \Rightarrow \quad \vec{u} = |\vec{u}|\,\hat{u}$$

* $\overleftrightarrow{AB}$ is the line of support of $\overrightarrow{AB}$

(iii) Coinitial vector :- Vectors having same initial point are called coinitial vector.

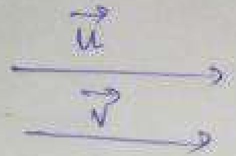
Ex :



'O' is the coinitial point

(iv) Parallel vector :- Two vectors are said to be parallel if their line of support are parallel.

Ex :



(v) Colinear vectors :- Vectors having same line of actions.

'OR'

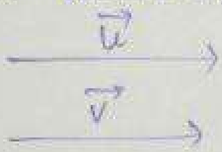
If they are parallel.

(vi) Coplaner vectors :- Vectors which lies on the same plane or parallel to vectors lying on the same plane.

(vii) Skew vectors :- Vectors which are not coplanar are called skew vectors.

(viii) Like vectors :- Vectors having same direction are called like vectors.

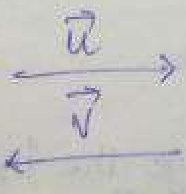
Ex :



~~or, otherwise they are called~~

(ix) Opposite vectors :- Vectors having opposite in direction are called opposite vectors.

Ex :



(x) Negative vector : Let $\vec{u}$ be a vector than negative vector of $\vec{u}$ is $-\vec{u}$ ha same magnitude but opposite direction.

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Multiplication of a vector by a scalar

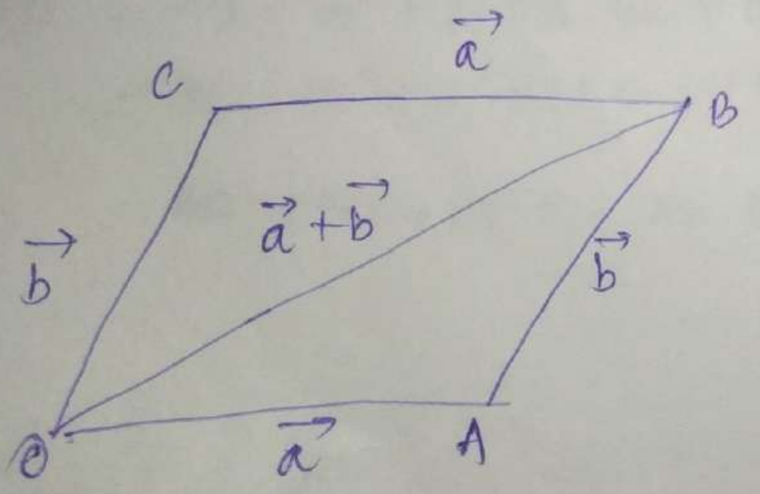
Let $\vec{u}$ be a vector. Let α be a scalar then multiplication of $\vec{u}$ by α is a vector $\alpha\vec{u}$ whose magnitude is $\alpha|\vec{u}|$ and direction :

(i) same as $\vec{u}$ if $\alpha > 0$.

(ii) same as $\vec{u}$ if $\alpha < 0$.

* If $\alpha = 0$ then $\alpha\vec{u} = \vec{0}$

Addition of two vectors (Parallelogram law of vector addition)



Let $\vec{a}$ and $\vec{b}$ two vectors represented by two sides of a parallelogram taken in order then their sum is represented by the diagonal of the parallelogram whose initial point is same as initial point of $\vec{a}$.

$$\vec{OA} = \vec{a}$$
$$\vec{AB} = \vec{b}$$

Then, $\vec{a} + \vec{b} = \vec{OA} + \vec{AB}$
$= \vec{OB}$

The vector $\vec{a} - \vec{b}$ is represented by the other diagonal of the parallelogram w initial point is the terminal point of $\vec{b}$.

$$\vec{OA} = \vec{a}$$
$$\vec{OC} = \vec{b}$$

Then, $\vec{a} - \vec{b} = \vec{OA} - \vec{OC}$
$= \vec{CA}$

Properties of vector addition

(i) Vector addition is commutative.
$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

(ii) Vector addition is associative
$$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$$

(iii) Vector addition is distributive
$$\alpha(\vec{a} + \vec{b}) = \alpha\vec{a} + \alpha\vec{b}$$
$$(\alpha + \beta)\vec{a} = \alpha\vec{a} + \beta\vec{a}$$

(iv) $(\alpha\beta)\vec{a} = \alpha(\beta\vec{a})$

(v) $1\vec{a} = \vec{a}$

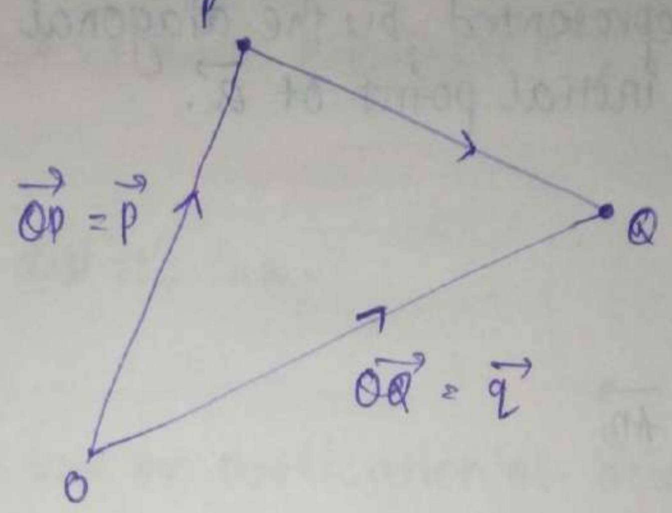
(vi) $0\vec{a} = \vec{0}$

(vii) Additive inverse

$$\vec{a} + (-\vec{a}) = 0$$
$$\vec{a} + \vec{b} = \vec{0}$$
$$\vec{b} = -\vec{a}$$

→ $\vec{a}$ is additive inverse of $\vec{b}$ and $\vec{b}$ is called additive inverse of $\vec{a}$.

Position vector



Let 'O' be a fixed point. Then the vector $\vec{OP}$ is called the position vector of point 'p' relative to 'O'. Similarly $\vec{OQ}$ is the position vector of point 'Q' relative to 'O'.

Here,

$$\vec{OP} + \vec{PQ} = \vec{OQ}$$

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$$= \text{Position vector of } Q - \text{position vector of } P$$

$$= \vec{q} - \vec{p}$$

~~Ex - If the position vectors of two given points~~

Resolution of a vector into components

$\vec{u}$ is a 2D Vector

$\vec{u} = x\hat{i} + y\hat{j}$

x, y are scalar components.

$x\hat{i}$, is vector component along x-axis

$y\hat{j}$, is vector component along y-axis.

If $\vec{u}$ is a 3D vector

$\vec{u} = x\hat{i} + y\hat{j} + z\hat{k}$

$x\hat{i}$, is vector component along x-axis.
$y\hat{j}$, is vector component along y-axis.
$z\hat{k}$, is vector component along z-axis.

Modulus

$\vec{u} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{u}| = \sqrt{x^2 + y^2 + z^2}$

Ex - Find modulus of $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$

Soln -
$$|\vec{a}| = \sqrt{1^2 + (-1)^2 + 3^2}$$
$$= \sqrt{1 + 1 + 9}$$
$$= \sqrt{11}$$

Unit vector

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{11}} = \frac{1}{\sqrt{11}}\hat{i} - \frac{1}{\sqrt{11}}\hat{j} + \frac{3}{\sqrt{11}}\hat{k}$$

Ex - Find the modulus and unit vector in the direction of the sum of the vectors $\hat{i}+4\hat{j}+2\hat{k}$, $3\hat{i}-3\hat{j}-2\hat{k}$ and $-2\hat{i}+2\hat{j}+6\hat{k}$.

Soln -

Let, $\vec{a} = \hat{i}+4\hat{j}+2\hat{k}$

$\vec{b} = 3\hat{i} - 3\hat{j} - 2\hat{k}$

$\vec{c} = -2\hat{i}+2\hat{j}+6\hat{k}$

Let $\vec{r} = \vec{a}+\vec{b}+\vec{c}$

$= (1+3-2)\hat{i} + (4-3+2)\hat{j} + (2-2+6)\hat{k}$

$\vec{r} = 2\hat{i}+3\hat{j}+6\hat{k}$

Now, $|\vec{r}| = \sqrt{2^2+3^2+6^2}$

$= \sqrt{4+9+36}$

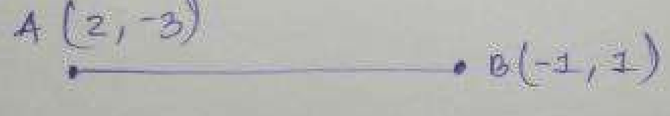
$= \sqrt{49} = 7$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$$

$$= \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

Ex - Find the vector joining the points $(2,-3)$ and $(-1,1)$. Find its magnitude and unit vector along the same direction. Also determine the scalar components and components vectors along the co-ordinate axis.

Soln -



$\vec{a}$ = Position vector of B - Position vector of A

$= (-1-2)\hat{i} + \{1-(-3)\}\hat{j}$

$= -3\hat{i}+4\hat{j}$

$|\vec{a}| = \sqrt{(-3)^2+(4)^2}$

$= \sqrt{9+16} = \sqrt{25}$

$= 5$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{-3\hat{i} + 4\hat{j}}{5} = \frac{-3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

So, Scalar components are $-\frac{3}{5}$ and $\frac{4}{5}$

Components vectors are $-\frac{3}{5}\hat{i}$ and $\frac{4}{5}\hat{j}$ along x-axis and y-axis respectively.

Ex- If the position vectors of two ~~points~~ given points A and B are $7\hat{i} + 3\hat{j} + \hat{k}$ and $2\hat{i} - 5\hat{j} + 4\hat{k}$ respectively. find the magnitude and direction of $\vec{AB}$

Soln- Position vector of $A = 7\hat{i} + 3\hat{j} + \hat{k}$

Position vector of $B = 2\hat{i} - 5\hat{j} + 4\hat{k}$

$\vec{AB}$ = Position vector of B - Position vector of A

$= (2-7)\hat{i} + (-5-3)\hat{j} + (4-1)\hat{k}$

$= -5\hat{i} - 8\hat{j} + 3\hat{k}$

$$|\vec{AB}| = \sqrt{(-5)^2 + (-8)^2 + (3)^2}$$

$$= \sqrt{25 + 64 + 9}$$

$$= \sqrt{98}$$

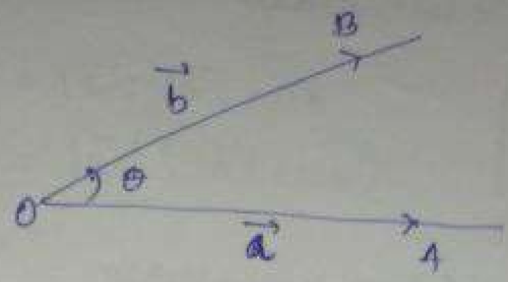
$$\hat{AB} = \frac{\vec{AB}}{|\vec{AB}|} = \frac{-5\hat{i} - 8\hat{j} + 3\hat{k}}{\sqrt{98}}$$

$$= \frac{-5}{\sqrt{98}}\hat{i} - \frac{8}{\sqrt{98}}\hat{j} + \frac{3}{\sqrt{98}}\hat{k}$$

Directions cosines are

$$= \left\langle \frac{-5}{\sqrt{98}}, \frac{-8}{\sqrt{98}}, \frac{3}{\sqrt{98}} \right\rangle$$

Scalar product or dot product



Let $\vec{a} = \vec{OA}$

$\vec{b} = \vec{OB}$

θ = inclination between $\vec{a}$ and $\vec{b}$

Define $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

The RHS of (1) is a scalar, so $\vec{a}\cdot\vec{b}$ is called scalar product.

$$\vec{b}\cdot\vec{a} = |\vec{b}|\,|\vec{a}|\cos\theta$$
$$= |\vec{a}|\,|\vec{b}|\cos\theta$$
$$= \vec{a}\cdot\vec{b}$$

$\Rightarrow$ scalar product / dot product is commutative.

* $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$$

This is the angle between two vectors.

* $\vec{a} \perp \vec{b}$, $\theta = 90^\circ$

Then,

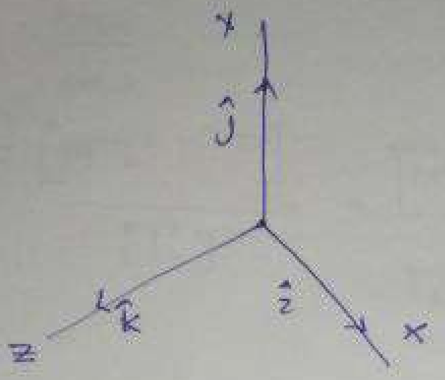
$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos 90^\circ$$
$$= 0$$

So, two vectors are perpendicular iff their dot product is zero.

* $\vec{a} \cdot \vec{a} = |\vec{a}| \, |\vec{a}| \cos\theta$
$= |\vec{a}|^2 \; 1$

Orthogonal unit vectors

$\hat{i}, \hat{j}, \hat{k}$



$\theta = 90°$

$\hat{i} \cdot \hat{j} = 0$

$\hat{j} \cdot \hat{k} = 0$

$\hat{k} \cdot \hat{i} = 0$

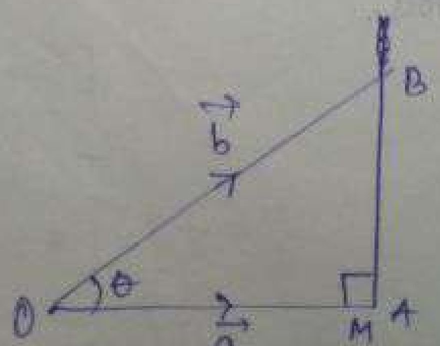
$\theta = 0$

* $\hat{i} \cdot \hat{i} = |\hat{i}|^2$
$= 1$

$\hat{j} \cdot \hat{j} = 1$

$\hat{k} \cdot \hat{k} = 1$

Geometrical meaning of dot product



$BM \perp OA$

$\vec{a} = \vec{OA}$

$\vec{b} = \vec{OB}$

θ = inclination between $\vec{a}$ and $\vec{b}$

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$
$$= |\vec{a}| \left(|\vec{b}| \cos\theta \right)$$

$$\cos\theta = \frac{OM}{OB}$$
$$OM = OB \cos\theta$$
$$= |\vec{b}| \cos\theta$$

$\Rightarrow \vec{a} \cdot \vec{b}$ = ~~modulus~~ a modulus of $\vec{a}$ × projection of $\vec{b}$ on $\vec{a}$

$$\boxed{\text{Projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}}$$

$$\vec{b} \cdot \vec{a} = |\vec{b}| \left(|\vec{a}| \cos\theta \right)$$

= modulus of $\vec{b}$ × projection of $\vec{a}$ on $\vec{b}$

$$\boxed{\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}}$$

Vector projection of $\vec{a}$ on $\vec{b}$

$$= \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \right) \hat{b}$$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \cdot \frac{\vec{b}}{|\vec{b}|}$$

$$= \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b}$$

Vector projection of $\vec{b}$ on $\vec{a}$

$$= \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

* If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\boxed{\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

Ex - find $\vec{a}\cdot\vec{b}$. If $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - 5\hat{k}$

$$\vec{a}\cdot\vec{b} = (2\times1) + (-1\times2) + (3\times(-5))$$
$$= 2 + (-2) + (-15)$$
$$= 2 - 2 - 15$$
$$= -15$$

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$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Scalar ~~product~~ projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

Scalar projection of $\vec{b}$ on $\vec{a}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

Vector projection of $\vec{a}$ on $\vec{b}$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

Vector projection of $\vec{b}$ on $\vec{a}$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

If $\vec{a} \perp \vec{b}$

$\rightarrow \vec{a}\cdot\vec{b} = 0$

$\hat{i}\cdot\hat{j} = 0$

$\hat{j}\cdot\hat{k} = 0$

$\hat{k}\cdot\hat{i} = 0$

Ex- Find the scalar and vector projections of the vector $2\hat{i} - 3\hat{j} - 6\hat{k}$ on the line joining the points $(3, 4, -2)$ and $(5, 6, -3)$.

Soln - Let $\vec{a} = 2\hat{i} - 3\hat{j} - 6\hat{k}$

Let $\vec{b}$ be the vector joining $(3,4,-2)$ and $(5,6,-3)$

$\Rightarrow \vec{b} = (5-3)\hat{i} + (6-4)\hat{j} + (-3+2)\hat{k}$

$= 2\hat{i} + 2\hat{j} - \hat{k}$

Now,

Scalar projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$$= \frac{(2\hat{i} - 3\hat{j} - 6\hat{k}) \cdot (2\hat{i} + 2\hat{j} - \hat{k})}{\sqrt{2^2 + 2^2 + (-1)^2}}$$

$$= \frac{(2 \times 2) + (-3) \times 2 + (-6) \times (-1)}{\sqrt{9}}$$

$$= \frac{4 - 6 + 6}{3} = \frac{4}{3}$$

Vector projection of $\vec{a}$ on $\vec{b}$

$$= \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right) \vec{b}$$

$$= \left(\frac{4}{3^2}\right)(2\hat{i} + 2\hat{j} - \hat{k})$$

$$= \frac{4}{9}(2\hat{i} + 2\hat{j} - \hat{k})$$

$$= \frac{8}{9}\hat{i} + \frac{8}{9}\hat{j} - \frac{4}{9}\hat{k}$$

Ans

Ex. Q. Find the scalar and vector projection of the vector $\hat{i}-\hat{j}-\hat{k}$ on $3\hat{i}+\hat{j}+3\hat{k}$.

Soln – Let $\vec{a} = \hat{i}-\hat{j}-\hat{k}$

$\vec{b} = 3\hat{i}+\hat{j}+3\hat{k}$

Now, scalar projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$$= \frac{(\hat{i}-\hat{j}-\hat{k})\cdot(3\hat{i}+\hat{j}+3\hat{k})}{\sqrt{3^2+1^2+3^2}}$$

$$= \frac{(1\times3)+(-1)\times1+(-1)\times3}{\sqrt{19}}$$

$$= \frac{3+(-1)+(-3)}{\sqrt{19}}$$

$$= \frac{3-1-3}{\sqrt{19}} = \frac{-1}{\sqrt{19}}$$

Vector projection of $\vec{a}$ on $\vec{b}$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

$$= \left(\frac{-1}{(\sqrt{19})^2}\right)(3\hat{i}+\hat{j}+3\hat{k})$$

$$= \frac{-1}{19}(3\hat{i}+\hat{j}+3\hat{k})$$

Q. Find the scalar product and angle between $\vec{a}$ and $\vec{b}$, $\vec{a} = (2,-2,1)$ and $\vec{b} = (0,2,4)$.

soln - $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = 2\hat{j} + 4\hat{k}$

$\vec{a}\cdot\vec{b} = (2\hat{i} - 2\hat{j} + \hat{k})(2\hat{j} + 4\hat{k})$

$= 2\times0 + (-2)\times2 + 1\times4$

$= 0 + (-4) + 4$

$= -4+4$

$= 0$

$\vec{a}\cdot\vec{b} = 0$

$\Rightarrow \theta = 90^\circ$

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ $\Rightarrow \theta = \cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$

$= \dfrac{0}{|\vec{a}||\vec{b}|} = 0$

$\theta = \cos^{-1}(0)$

$= 90^\circ$ $= \cos^{-1}(\cos 90^\circ)$

Q. Find the value of λ so that the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

$\vec{a} = 2\hat{i} - \hat{j} - \hat{k}$

$\vec{b} = \lambda\hat{i} + \hat{j} + 5\hat{k}$

soln - If $\vec{a} \perp \vec{b}$

$\Rightarrow \vec{a}\cdot\vec{b} = 0$

$\Rightarrow 2\lambda - 1 - 5 = 0$

$\Rightarrow 2\lambda = 6$

$\Rightarrow \lambda = \dfrac{6}{2} = 3$

(ii) $\vec{a} = \hat{i} + \hat{j} + \lambda \hat{k}$

$\vec{b} = 4\hat{i} - 3\hat{k}$

$\Rightarrow$ If $\vec{a} \perp \vec{b}$

$= \vec{a} \cdot \vec{b} = 0$

$\Rightarrow 4 + 0 - 3\lambda$

$\Rightarrow 3\lambda = 4$

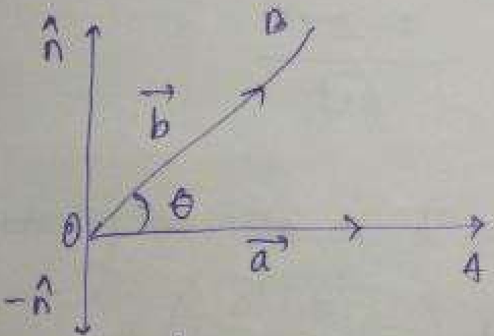
$\Rightarrow \lambda = \frac{4}{3}$

Assignment

12(b)

Q.1 - Q.6

Vector product or cross product



Let θ be the inclination between $\vec{a}$ and $\vec{b}$

Define

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$

here, $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$

so, $\vec{a}, \vec{b}, \hat{n}$ form a right handed system.

so, $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$

$\therefore$ A unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$$= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$\vec{b} \times \vec{a} = |\vec{b}||\vec{a}| \sin\theta \, (-\hat{n})$$

$$= |\vec{a}||\vec{b}| \sin\theta \, (-\hat{n})$$

$$= -\left(|\vec{a}||\vec{b}| \sin\theta \, \hat{n}\right)$$

$$= -\cancel{a}\left(\vec{a} \times \vec{b}\right)$$

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$$

= Cross product is not commutative.

COMMUNICATIVE ENGLISH

* $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ have same magnitude but opposite direction.
* $\vec{a} \perp \vec{b}$, $\theta = 90°$

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin 90°$

$= |\vec{a}||\vec{b}| \hat{n}$

* If $\vec{a} \perp \vec{b}$ and $\vec{a}, \vec{b}$ are unit vectors

$\vec{a} \times \vec{b} = 1 \cdot 1 \cdot \hat{n}$

$= \hat{n}$

* $\hat{i} \perp \hat{j} \perp \hat{k}$

$\hat{i} \times \hat{j} = \hat{k}$

$\hat{j} \times \hat{i} = -\hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$

$\hat{k} \times \hat{j} = -\hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$

$\hat{i} \times \hat{k} = -\hat{j}$

$\hat{i} \times \hat{i} = |\hat{i}||\hat{i}| \sin 0 \, \hat{n}$

$= \vec{0}$

$\hat{j} \times \hat{j} = \vec{0}$

$\hat{k} \times \hat{k} = \vec{0}$

* $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$

$$\boxed{\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}}$$

Properties of cross product

(i) Cross product is not commutative.

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$$

(ii) It is associative with respect to a scalar.

$$\alpha(\vec{a} \times \vec{b}) = (\alpha\vec{a}) \times \vec{b}$$
$$= \vec{a} \times (\alpha\vec{b})$$

(iii) Vector product is distributive.

$$\vec{a} \times (\vec{b}+\vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

* $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\vec{a} \times \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$
$$= a_1\hat{i} \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{j} \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k}(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$
$$= a_1b_2\hat{k} - a_1b_3\hat{j} - a_2b_1\hat{k} + a_2b_3\hat{i} + a_3b_1\hat{j} - a_3b_2\hat{i}$$
$$= \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1)$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Ex. Find a unit vector perpendicular to each of the vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$. Also find the sine of angle between the two vectors.

Soln – $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{b} = 3\hat{i} + 4\hat{j} - \hat{k}$

A unit vector perpendicular to $\vec{a}$ & $\vec{b}$

$$= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$\hat{i}$ (23)

$\hat{j}$ (13)

$\hat{k}$ (12)

$$= \hat{i}(1-4) - \hat{j}(-2-5) + \hat{k}(8+3)$$

$$= -3\hat{i} + 5\hat{j} + 11\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-3)^2 + (5)^2 + (11)^2}$$

$$= \sqrt{9 + 25 + 121}$$

$$= \sqrt{155}$$

So, unit vector perpendicular to $\vec{a}$ & $\vec{b}$

$$= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$= \frac{-3\hat{i} + 5\hat{j} + 11\hat{k}}{\sqrt{155}}$$

$$= \frac{-3}{\sqrt{155}}\hat{i} + \frac{5}{\sqrt{155}}\hat{j} + \frac{11}{\sqrt{155}}\hat{k}$$

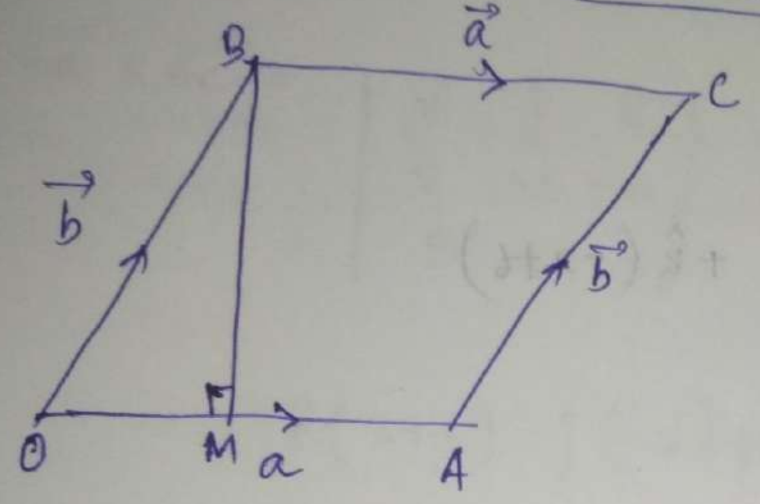
Sine of the angle between $\vec{a}$ & $\vec{b}$

$$= \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

$$= \frac{\sqrt{155}}{\sqrt{(2)^2 + (-1)^2 + (1)^2}\ \sqrt{(3)^2 + (4)^2 + (-1)^2}}$$

$$= \frac{\sqrt{155}}{\sqrt{6}.\sqrt{36}} = \frac{\sqrt{155}}{\sqrt{156}}$$

Geometrical meaning of cross product



$\vec{OA} = \vec{a}$

$\vec{OB} = \vec{b}$

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$

$= |\vec{a}|\,BM\,\hat{n}$

$|\vec{a} \times \vec{b}| = |\vec{a}|\,BM$

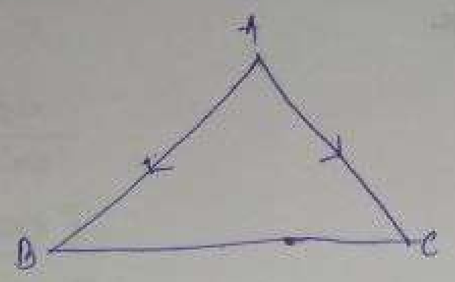
= Area of parallelogram with sides $\vec{a}$ and $\vec{b}$

$\sin\theta = \frac{BM}{OB}$

$BM = |\vec{OB}|\sin\theta$

$= |\vec{b}|\sin\theta$

Hence, $\vec{a} \times \vec{b}$ is a vector whose magnitude is equal to the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$.



The area of ΔABC

$$= \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right|$$

Ex- Obtain the area of the parallelogram whose sides are vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $-3\hat{i}-2\hat{j}+\hat{k}$.

soln - $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$

$\vec{b} = -3\hat{i}-2\hat{j}+\hat{k}$

Area of parallelogram with sides $\vec{a}$ & $\vec{b}$ = $|\vec{a}\times\vec{b}|$

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix}$

$$\vec{a}\times\vec{b} = \hat{i}(2+6) - \hat{j}(1+9) + \hat{k}(-2+6)$$
$$= 8\hat{i} - 10\hat{j} + 4\hat{k}$$

Area = $|\vec{a}\times\vec{b}|$

$= \sqrt{(8)^2+(-10)^2+(4)^2}$

$= \sqrt{64+100+16}$

$= \sqrt{180}$ squnit

Q. obtain the area of the parallelogram whose sides are vectors $(1,-3,1)$ $(1,1,1)$

soln= $\vec{a} = \hat{i}-3\hat{j}+\hat{k}$

$\vec{b} = \hat{i}+\hat{j}+\hat{k}$

Area of parallelogram = $|\vec{a}\times\vec{b}|$

$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$\Rightarrow \vec{a} \times \vec{b} = \hat{i}(-3-1) - \hat{j}(1-1) + \hat{k}(1+3)$

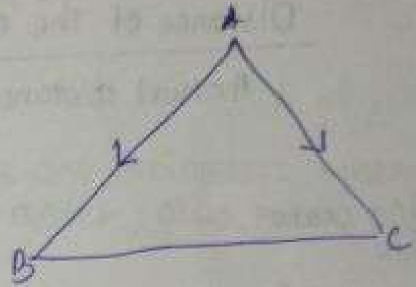
$= -4\hat{i} + 4\hat{k}$

$|\vec{a} \times \vec{b}| = \sqrt{(-4)^2 + (4)^2}$

$= \sqrt{32}$ sq. unit.

Q. Calculate the area of ΔABC by vector method $A(1,2,4)$, $B(3,1,-2)$ $C(4,3,1)$.

Soln - Area of $\Delta ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|$



$\vec{AB} = (3-1)\hat{i} + (1-2)\hat{j} + (-2-4)\hat{k}$

$= 2\hat{i} - \hat{j} - 6\hat{k}$

$\vec{AC} = (4-1)\hat{i} + (3-2)\hat{j} + (1-4)\hat{k}$

$= 3\hat{i} + \hat{j} - 3\hat{k}$

Now, $\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -6 \\ 3 & 1 & -3 \end{vmatrix}$

$= \hat{i}(3+6) - \hat{j}(-6+18) + \hat{k}(2+3)$

$= 9\hat{i} - 12\hat{j} + 5\hat{k}$

Now area of $\Delta ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

$= \frac{1}{2}\sqrt{(9)^2 + (-12)^2 + (5)^2}$

$= \frac{1}{2}\sqrt{81 + 144 + 25}$

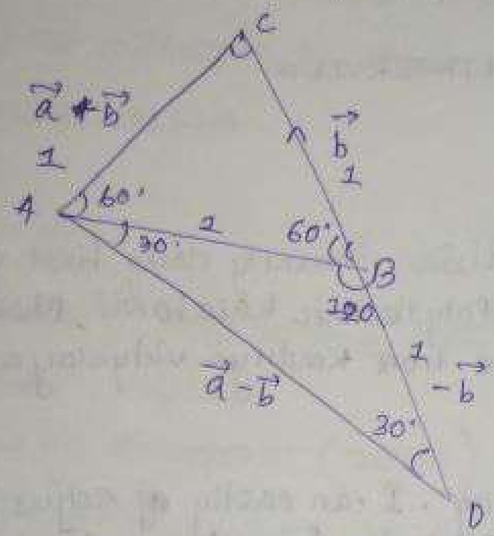
$= \frac{1}{2} \times \sqrt{250} = \frac{5\sqrt{10}}{2}$

Q: The sum of two unit vector is a unit vector. Then show that magnitude of their difference is $\sqrt{3}$.

soln - $\vec{a}, \vec{b}$ is a unit vector

$\Rightarrow \vec{a}+\vec{b}$ is a unit vector

To show that $|\vec{a}-\vec{b}| = \sqrt{3}$



In $\triangle ABC$,

$\overrightarrow{AB} = \vec{a}$

$\overrightarrow{BC} = \vec{b}$

$\overrightarrow{AC} = \vec{a}+\vec{b}$

But, $|\vec{a}| = |\vec{b}| = |\vec{a}+\vec{b}| = 1$

so, $\triangle ABC$ is an equilateral triangle

$\Rightarrow$ Measurement of $\angle CAB$ = measurement of $\angle ABC$ = measurement of $\angle BCA = 60°$

Now produce BD such that $\overrightarrow{BD} = -\vec{b}$

Join AD

Now, $|\overrightarrow{AB}| = |\overrightarrow{BD}| = 1$, $\angle ABD = 120°$

so, $\triangle ABD$ is a isoscales triangle.

$\Rightarrow \angle BAD = \angle BDA = 30°$

$\overrightarrow{AD} = \vec{a}-\vec{b}$, $\overrightarrow{CD} = 1+1 = 2$

Now, consider $\triangle ADC$

$m\angle CAD = 90°$

$\Rightarrow \Delta ADC$ is a right angled triangle

so, $|\vec{AC}|^2 + |AD|^2 = |\vec{CD}|^2$

$\Rightarrow 1 + |\vec{a} - \vec{b}|^2 = 2^2$

$\Rightarrow |\vec{a} - \vec{b}|^2 = (2)^2 - (1)^2 = 3$

$\Rightarrow |\vec{a} - \vec{b}| = \sqrt{3}$

EXERCISE 12(c)

Q. Calculate the area of ΔABC by vector method where $A(1,2,4)$, $B(3,1,-$ $C(4,3,1)$

Soln - $\vec{AB} = (3-1)\hat{i} + (1-2)\hat{j} + (-2-4)\hat{k}$

$= 2\hat{i} - \hat{j} - 6\hat{k}$

$\vec{AC} = (4-1)\hat{i} + (3-2)\hat{j} + (1-4)\hat{k}$

$= 3\hat{i} + \hat{j} - 3\hat{k}$

~~Area of $\Delta ABC = \frac{1}{2}$~~ $|\vec{AB} \times \vec{AC}|$

Now, $\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -6 \\ 3 & 1 & -3 \end{vmatrix}$

$= \hat{i}(3+6) - \hat{j}(-6+18) + \hat{k}(2+3)$

$= 9\hat{i} - 12\hat{j} + 5\hat{k}$

Now area of $\Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|$

$= \frac{1}{2}\sqrt{(9)^2 + (-12)^2 + (5)^2}$

$= \frac{1}{2}\sqrt{81 + 144 + 25}$

$= \frac{1}{2} \times \sqrt{250}$

$= 5\sqrt{10}$

Limit & Continuity

Cartesian product

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

Function

It is a special type of relation.

$f : A \to B$

① $Dom_f = A$

② $(x, y) \in f$

$(x, z) \in f$

$\Rightarrow y = z$

It is not one - many relation.

$$\underset{\text{Independent}}{\overset{\text{dependent}}{y}} = f(x)$$

$y = mx + c$ — m, c: Constant

Types of function

① Constant function

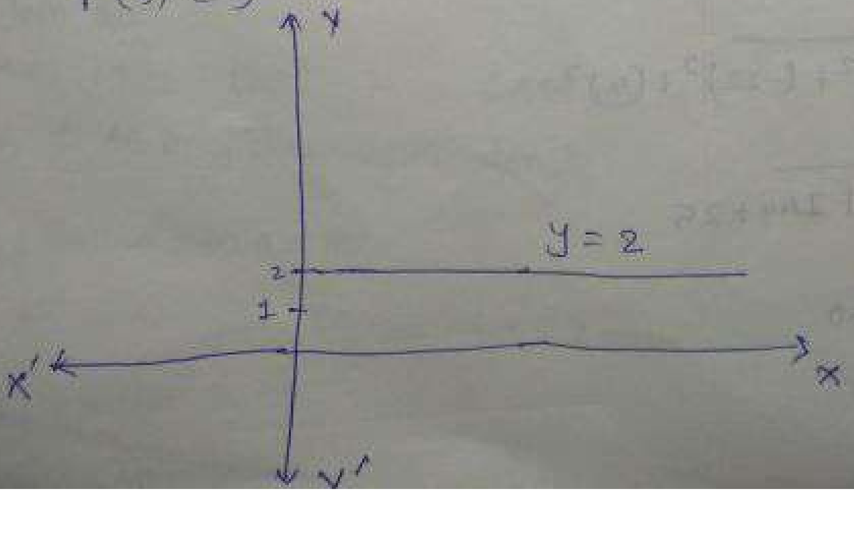
A function $f : A \to R$ is said to be a constant function if $f(x) = k$, for k be any real number $\forall x \in A$.

Ex - $f(x) = 2$

$f(1) = 2$

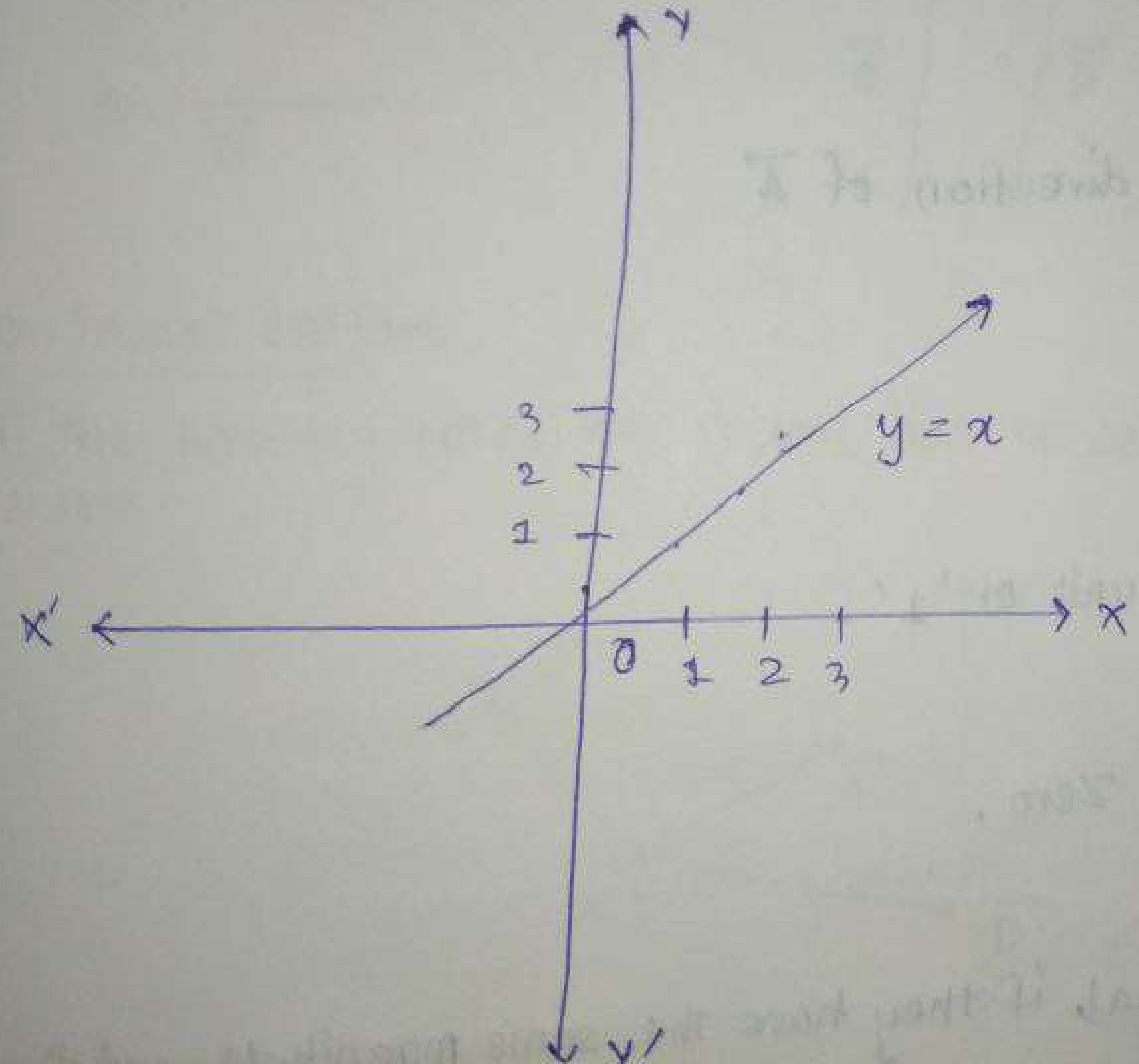
$f(2) = 2$

$f(3) = 3$



(2) Identity function

Let $f : A \to A$ where $A \in R$ such that $f(x) = x, \forall x \in A$ here f is called an identity function.



Modulus function / Absolute value function

$f : R \to R$

defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

Ex - $|0| = 0$

$|1| = 1$

$|2| = 2$

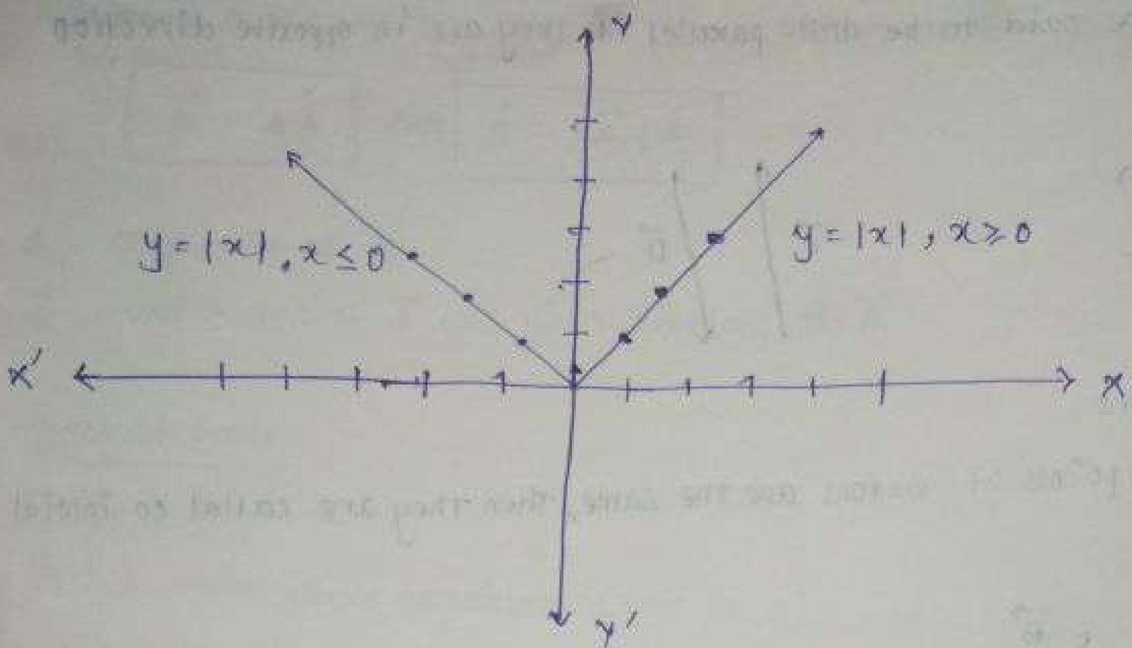
$|15| = 15$

$|-1| = -(-1) = 1$

$|-25| = -(-25) = 25$

$|-100| = -(-100) = 100$

Graph representation

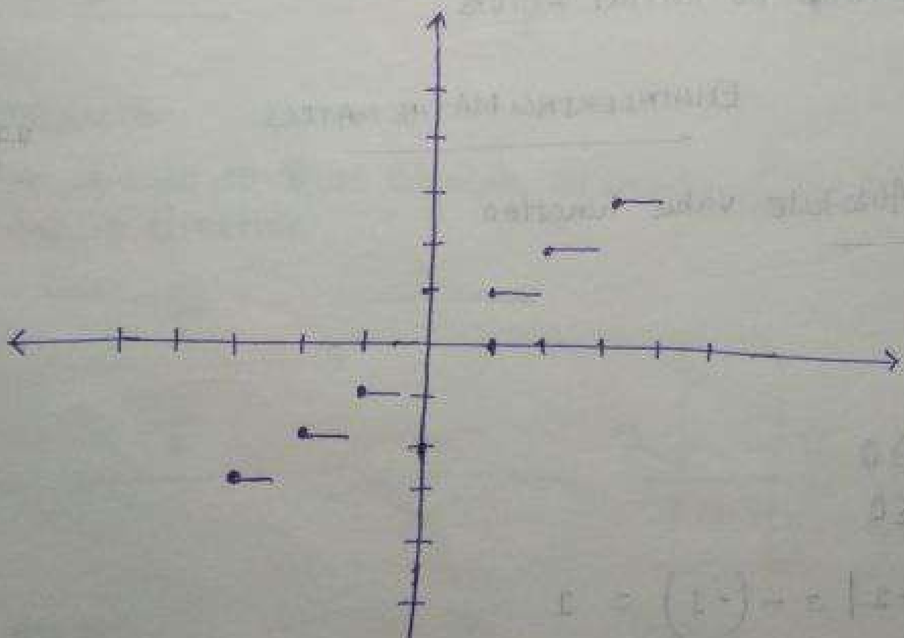


Greatest integer function $[x]$

A function $f : R \to Z$ is said to be greatest integer function if
$[x] = n$ if $n \leq x < n+1$

"OR"

$[x]$ is the greatest integer not exceeding x.



This is also known as step function.

Signum function ($sgn\, x$)

The signum function in R is defined by $sgn\, x = \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

$x = 1$, $sgn\, x = \frac{1}{|1|} = \frac{1}{1} = 1$

$x = 5$, $sgn\,(x) = \frac{5}{|5|} = \frac{5}{5} = 1$

$x = 100$, $sgn\, x = \frac{100}{|100|} = \frac{100}{100} = 1$

$x = -1$, $sgn\, x = \frac{-1}{|-1|} = \frac{-1}{1} = -1$

$x = -2$, $sgn\, x = \frac{-2}{|-2|} = \frac{-2}{2} = -1$

$x = -5$, $sgn\, x = \frac{-5}{|-5|} = \frac{-5}{5} = -1$

Range set of $sgn\, x = \{-1, 0, 1\}$

ENGINEERING MECHANICS

... theorem the resultant of force

$$\vec{b} \times \vec{a} = |\vec{b}||\vec{a}| \sin\theta \, (-\hat{n})$$

$$= |\vec{a}||\vec{b}| \sin\theta \, (-\hat{n})$$

$$= -\left(|\vec{a}||\vec{b}| \sin\theta \, \hat{n}\right)$$

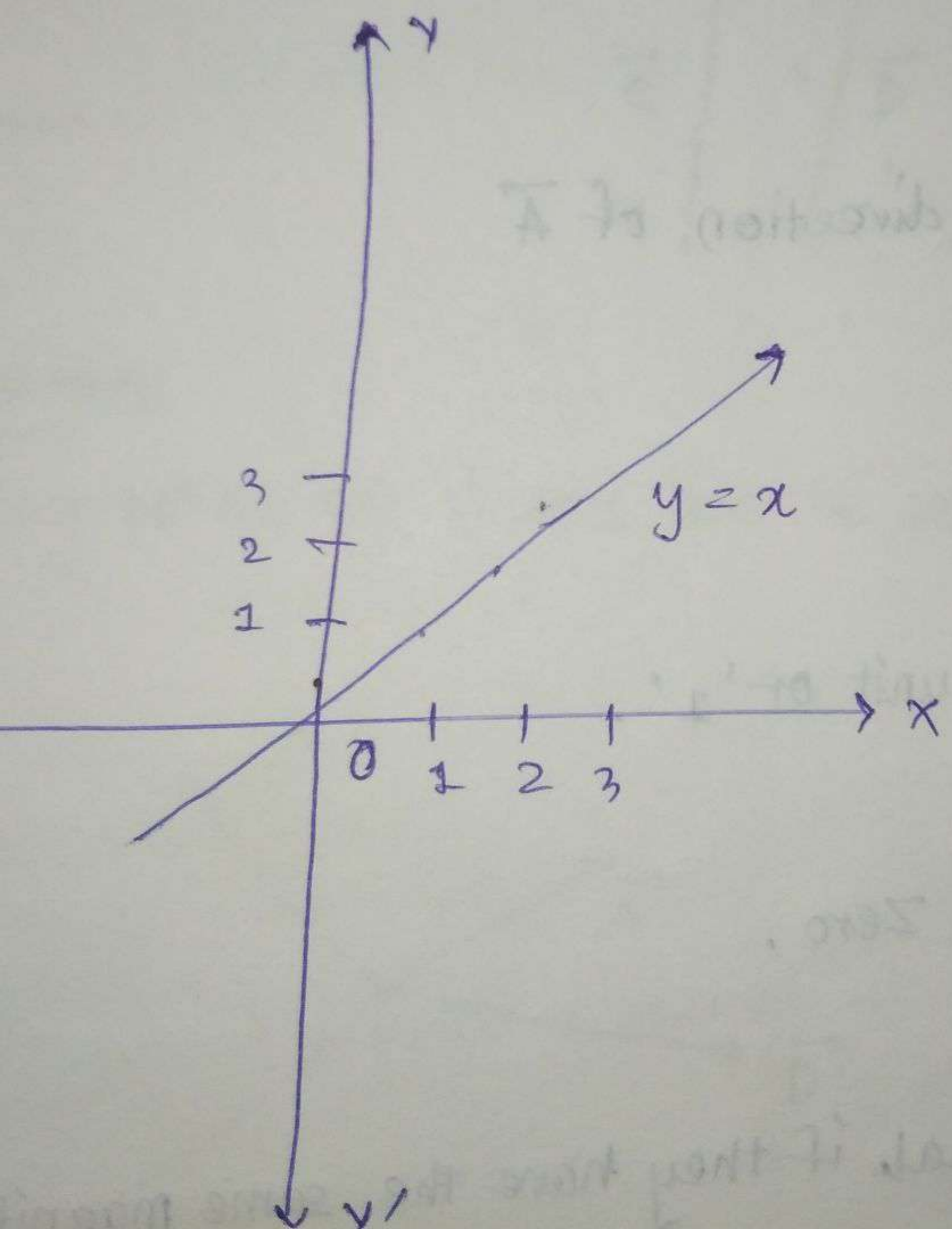
$$= -\left(\vec{a} \times \vec{b}\right)$$

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$$

= Cross product is not commutative

tion

where $A \subseteq R$ such that $f(x) = x, \forall x \in A$ h
ion.



Modulus function / Absolute value function

$f : R \to R$

defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

Ex - $|0| = 0$

$|1| = 1$

$|2| = 2$

$|15| = 15$

$|-1| = -(-1) = 1$

$|-25| = -(-25) = 25$

$|-100| = -(-100) = 100$

Absolute value function / Modulus function

Trigonometric function

Sin : $\mathbb{R} \longrightarrow [-1, 1]$

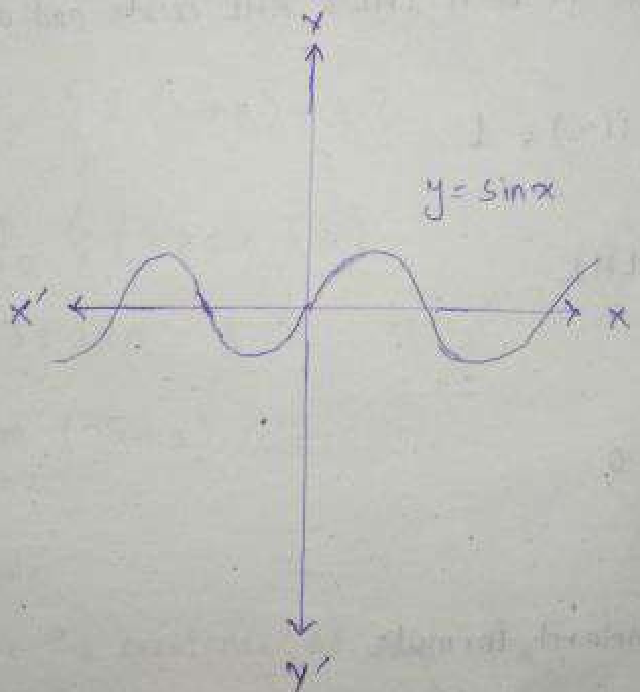
Cosine : $\mathbb{R} \rightarrow [-1, 1]$

tan : $\mathbb{R} - \{(2n+1)\frac{\pi}{2}\} \rightarrow \mathbb{R}$

cot : $\mathbb{R} - \{n\pi\} \longrightarrow \mathbb{R}$

Sec : $\mathbb{R} - \{(2n+1)\frac{\pi}{2}\} \longrightarrow (-\infty, -1] \cup [1, \infty)$

Cosec : $\mathbb{R} - \{n\pi\} \longrightarrow (-\infty, -1] \cup [1, \infty)$



Limit of a function

A no 'l' is said to be the limit of a function $f(x)$ as x tends to a.

i.e $\lim_{x \to a} f(x) = l$

if for $\epsilon > 0$ there exists $\delta > 0$ depending on ϵ such that

$0 < |x-a| < \delta \Leftrightarrow |f(x) - l| < \epsilon$

Concept of closeness neighbourhood

$x \in (a, b)$

$a < x < b$

$a-\delta$ $a+\delta$

a b

Left hand limit (LHL)

A no. 'l_1' is said to LHL of $f(x)$ as x tends to a

i.e $\lim_{x \to a} f(x) = l_1$ if for any $\varepsilon > 0$, there exists $\delta > 0$ depending on ε such that $a - \delta < x < a \Rightarrow |f(x) - l_1| < \varepsilon$

Right hand limit (RHL)

A no. 'l_2' is said to RHL of $f(x)$ as x tends to a

i.e $\lim_{x \to a+} f(x) = l_2$ if for any $\varepsilon > 0$ there exists $\delta > 0$ depending on ε such that $a < x < a + \delta \Rightarrow |f(x) - l_2| < \varepsilon$

Existence of a limit

A limit $\lim_{x \to a} f(x)$ exists if both LHL & RHL exists and are equal.

i.e $\lim_{x \to a^-} f(x) = \lim_{x \to a+} f(x) = l$

Procedure to evaluate a LHL

1. Write $\lim_{x \to a^-} f(x)$
2. Put $x = a - h$ and $h \to 0$
3. $\lim_{h \to 0} f(a-h)$
4. Then evaluate using standard formula

Procedure to evaluate RHL

1. Write $\lim_{x \to a+} f(x)$
2. Put $x = a + h$ and $h \to 0$
3. $\lim_{h \to 0} f(a+h)$
4. then evaluate the limit using standard formula.

Example

Examine the existence of $\lim_{x \to 1} (2x+1)$

Soln - LHL

$$\lim_{x \to 1^-} 2x+1$$

$$= \lim_{h \to 0} \{2(1-h)+1\}$$

$$= \lim_{h \to 0} (2-2h+1)$$

$$= \cancel{3+2\times 0 =} \; 3-2\times 0 = 3$$

RHL

$$\lim_{x \to 1^+} 2x+1$$

$$\Rightarrow \lim_{h \to 0} \{2(1+h)+1\}$$

$$\Rightarrow \lim_{h \to 0} (2+2h+1)$$

$$\Rightarrow 3+2\times 0 = 3$$

So, $\lim_{x \to 1} (2x+1)$ exists.

Example

Examine the existence of $\lim_{x \to n} [x]$

Soln -

LHL

$$\lim_{x \to n^-} [x]$$

$$= \lim_{h \to 0} [n-h]$$

$$= n-1$$

RHL

$$\lim_{x \to n^+} [x]$$

$$= \lim_{h \to 0} [n+h]$$

$$= n$$

LHL $\neq$ RHL

so, $\lim_{x \to n} [x]$ does not exists.

Ex - $\lim_{x \to 0} [x]$

soln -

LHL

$$\lim_{x \to 0^-} [x]$$

$$= \lim_{h \to 0} [0-h]$$

$$= -1$$

RHL

$$\lim_{x \to 0^+} [x]$$

$$= \lim_{h \to 0} [0+h]$$

$$= 0$$

LHL $\neq$ RHL

so, $\lim_{x \to 0} [x]$ does not exists

RHL

$$\lim_{x\to 0^+} \frac{x}{|x|}$$

$$= \lim_{h\to 0} \frac{0+h}{|0+h|}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}}$$

$$= 1$$

$\because$ LHL $\neq$ RHL

So, $\lim_{x\to 0}$ Sgn x does not exist.

Q. Examine the existence of limit at $x=0$

$$f(x) = \begin{cases} \dfrac{x-|x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

21/4/22

soln -

LHL

$$\lim_{x\to 0^-} \frac{x-|x|}{x}$$

$$\Rightarrow \lim_{h\to 0} \frac{(0-h)-|0-h|}{0-h}$$

$$\Rightarrow \lim_{h\to 0} \frac{-h-h}{-h}$$

$$\Rightarrow \lim_{h\to 0} \frac{-2\cancel{h}}{-\cancel{h}}$$

$$\Rightarrow 2$$

RHL

$$\lim_{x\to 0^+} \frac{x-|x|}{x}$$

$$\Rightarrow \lim_{h\to 0} \frac{(0+h)-|0+h|}{0+h}$$

$$\Rightarrow \lim_{h\to 0} \frac{h-h}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{0}{h}$$

$$\Rightarrow 0$$

LHL $\neq$ RHL

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Laws of limit

Let f and g be two functions such that $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$.

(i) $\lim_{x \to a} \left(f(x) + g(x) \right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

$$= l + m$$

Limit of sum of two functions is equal to sum of their limits.

(ii) $\lim_{x \to a} \left\{ f(x) - g(x) \right\} = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

$$= l - m$$

The limit of difference of two functions is equal to the difference of their limits.

(iii) $\lim_{x \to a} \left\{ k f(x) \right\} = k \lim_{x \to a} f(x)$

$$= kl$$

(iv) $\lim_{x \to a} \left\{ f(x) - g(x) \right\} = \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$

$$= lm$$

(v) $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{l}{m}, \quad m \neq 0$

Ex $\lim_{x \to 1} \left(\sqrt{x} + x + \frac{1}{\sqrt{x}} \right)$

$$= \lim_{x \to 1} \sqrt{x} + \lim_{x \to 1} + \lim_{x \to 1} \frac{1}{\sqrt{x}}$$

$= \sqrt{1} + 1 + \frac{1}{\sqrt{1}}$

$= 1 + 1 + 1 = 3$ | Ans

Example

$$\lim_{x \to 1} (17\sqrt{x})$$

$= 17 \lim_{x \to 1} \sqrt{x}$

$= 17 \cdot \sqrt{1}$

$= 17 \cdot 1$

$= 17$

Example

$$\lim_{x \to 1} \left(\frac{x + \sqrt{x}}{2x + 1} \right)$$

$$= \frac{\lim_{x \to 1} (x + \sqrt{x})}{\lim_{x \to 1} (2x + 1)} = \frac{\lim_{x \to 1} x + \lim_{x \to 1} \sqrt{x}}{\lim_{x \to 1} 2x + \lim_{x \to 1} 1}$$

$$= \frac{1 + \sqrt{1}}{2 \cdot 1 + 1}$$

$$= \frac{1 + \sqrt{1}}{2 + 1}$$

$$= \frac{2}{3}$$

Methods of evaluation of limits

We shall deal with the problems on

1) ~~Algebraic~~ Algebraic limit
2) Trigonometric limit
3) Exponential and logarithmic limits

Methods of evaluation of Algebric limit

1. Direct substitution method
2. factorisation method
3. Rationalisation method
4. Using some standard results
5. Evaluation of limits when $x \to \infty$ (limits at infinity)

Direct substitution method

In this case we shall directly substitute the value of x to evaluate the limit.

Indeterminate form
$\frac{0}{0}, \frac{\infty}{\infty}, \frac{\infty}{0}, \frac{1}{0}$

Example

$$\lim_{x \to 1} (1 + 2x - 2x^2 + 4x^3 - 5x^4)$$

Soln -

$$\lim_{x \to 1} 1 + \lim_{x \to 1} 2x - \lim_{x \to 1} 2x^2 + \lim_{x \to 1} 4x^3 - \lim_{x \to 1} 5x^4$$

$$= 1 + 2 \lim_{x \to 1} x - 2 \lim_{x \to 1} x^2 + 4 \lim_{x \to 1} x^3 - 5 \lim_{x \to 1} x^4$$

$$= 1 + 2 \cdot 1 - 2 \cdot 1^2 + 4 \cdot 1^3 - 5 \cdot 1^4$$

$$= 1 + \cancel{2} - \cancel{2} + 4 - 5$$

$$= 0$$

Example

$$\lim_{x \to 1} (2x^2 - 1)$$

Soln -

$$= 2 \cdot 1^2 - 1$$

$$= 2 - 1 = 1$$

Example

Evaluate $\lim_{x \to 2} \frac{x-2}{x^4-16}$

Soln = $\frac{0}{0}$ form

Now, using factorisation method | $a^4 b^4 = (a-b)(a+b)(a^2+b^2)$

$\lim_{x \to 2} \frac{x-2}{x^4-16}$

$\Rightarrow \lim_{x \to 2} \frac{\cancel{x-2}}{(\cancel{x-2})(x+2)(x^2+2^2)}$

$\Rightarrow \lim_{x \to 2} \frac{1}{(x+2)(x^2+2^2)}$

$\Rightarrow \lim_{x \to 2} \frac{1}{(2+2)(2^2+4)}$

$\Rightarrow \lim_{x \to 2} \frac{1}{(4)(8)}$

$\Rightarrow$ ~~Lt~~ $\frac{1}{32}$

Factorisation Method

$$\lim_{x \to 3} \frac{x^2-9}{x-3}$$ (Using direct substitution method)

$= \frac{0}{0}$ (Indeterminate form)

Now, Using factorisation method

$$\Rightarrow \lim_{x \to 3} \frac{x^2-3^2}{x-3}$$

$$\Rightarrow \lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$\Rightarrow \lim_{x \to 3} (x+3)$$

Apply limit

$\Rightarrow 3+3$

$\Rightarrow 6$ Ans

Example

$$\lim_{x \to 1} \frac{x^3-1}{x-1}$$

$= \frac{0}{0}$

Now using factorisation method

$$= \lim_{x \to 1} \frac{x^3-1^3}{x-1}$$ $a^3-b^3 = (a-b)(a^2+ab+b^2)$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x^2+x+1)$$

Apply limit

$= 1^2+1+1$

$= 3$

ENGINEERING MATHEMATICS

21/4/22

Q. Examine the existence of limit

$$\lim_{x \to 0} \operatorname{sgn} x$$

Soln - $f(x) = \operatorname{sgn}(x) \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

LHL

$$\lim_{x \to 0^-} \frac{x}{|x|}$$

$$= \lim_{h \to 0} \frac{0-h}{|0-h|}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}}$$

$$= -1$$

ENGINEERING MATHEMATICS

25/4/22

Q. $\lim_{x \to 0} \frac{(3+x)^3 - 27}{x}$

= Putting $x \to 0$ $\frac{(3+0)^3 - 27}{x}$

$= \frac{27-27}{0} = \frac{0}{0}$ (Indeterminate form)

Now, using factorisation method

$\lim_{x \to 0} \frac{(3+x)^3 - 3^3}{x}$

$\Rightarrow \lim_{x \to 0} \frac{(\cancel{3}+x-\cancel{3})\{(3+x)^2 + (3+x)3 + 3^2\}}{x}$

$$\Rightarrow \lim_{x \to 0} \frac{\cancel{x}\{(3+x)^2 + 3(3+x) + 9\}}{\cancel{x}}$$

Apply limit

$(3+0)^2 + 3(3+0) + 9$

$= 9 + 9 + 9$

$= 27$ Ans

Q. $\lim_{x \to 3} \frac{x^2 + 2x - 15}{x^2 - x - 6}$

$$\Rightarrow \lim_{x \to 3} \frac{x^2 + 5x - 3x - 15}{x^2 + 2x - 3x - 6}$$

$$\Rightarrow \lim_{x \to 3} \frac{x(x+5) - 3(x+5)}{x(x+2) - 3(x+2)}$$

$$\Rightarrow \lim_{x \to 3} \frac{(x+5)\cancel{(x-3)}}{(x+2)\cancel{(x-3)}}$$

Apply limit

$$\text{limit} = \frac{3+5}{3+2} = \frac{8}{5} \text{ Ans}$$

Evaluation of limit by Rationalisation method

By multiplying and dividing with conjugate

Ex. $\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}$

$$= \lim_{x \to 0} \frac{x(\sqrt{x+1} + 1)}{(\sqrt{x+1} - 1)(\sqrt{x+1} + 1)}$$

$$= \lim_{x \to 0} \frac{x(\sqrt{x+1} + 1)}{(\sqrt{x+1})^2 - 1^2}$$

$$= \lim_{x \to 0} \frac{x(\sqrt{x+1}+1)}{x+\cancel{1}-\cancel{1}}$$

$$= \lim_{x \to 0} \frac{\cancel{x}(\sqrt{x+1}+1)}{\cancel{x}}$$

Apply the limit

$= \sqrt{0+1} + 1$

$= 1 + 1 = 2$ Ans

Q. $\lim_{x \to 25} \frac{5-\sqrt{x}}{25-x}$

$$\Rightarrow \lim_{x \to 25} \frac{(5-\sqrt{x})(5+\sqrt{x})}{(25-x)(5+\sqrt{x})}$$

$$\Rightarrow \lim_{x \to 25} \frac{5^2-(\sqrt{x})^2}{(25-x)(5+\sqrt{x})}$$

$$\Rightarrow \lim_{x \to 25} \frac{\cancel{25-x}}{(\cancel{25-x})(5+\sqrt{x})}$$

$$\Rightarrow \lim_{x \to 25} \frac{1}{5+\sqrt{x}}$$

Apply limit

$$\frac{1}{5+\sqrt{25}} = \frac{1}{5+5}$$

$$= \frac{1}{10}$$

ENGINEERING MATHEMATICS 27/4/22

Limit at infinity $\left\{ \lim\limits_{x \to \infty} f(x) \right\}$

Ex - $\lim\limits_{x \to \infty} \dfrac{3x^2+4x-1}{2x^2-3x+5}$

Divide numerator and denominator by x^2

$$\Rightarrow \lim_{x \to \infty} \frac{\frac{3x^2}{x^2} + \frac{4x}{x^2} - \frac{1}{x^2}}{\frac{2x^2}{x^2} - \frac{3x}{x^2} + \frac{5}{x^2}}$$

$$\Rightarrow \lim_{x \to \infty} \frac{3 + \frac{4}{x} - \frac{1}{x^2}}{2 - \frac{3}{x} + \frac{5}{x^2}}$$

Now apply limit

$$\frac{3 + \cancel{\frac{4}{\infty}}^{\,0} - \cancel{\frac{1}{\infty^2}}^{\,0}}{2 - \cancel{\frac{3}{\infty}}^{\,0} + \cancel{\frac{5}{\infty^2}}^{\,0}}$$

$$= \frac{3}{2}$$ Ans

Ex - $\lim_{x \to \infty} \dfrac{2x+1}{3x-2}$

Divide numerator and denominator by x

$\Rightarrow \lim_{x \to \infty} \dfrac{\frac{2x}{x} + \frac{1}{x}}{\frac{3x}{x} - \frac{2}{x}}$

~~[illegible]~~

$= \lim_{x \to \infty} \dfrac{2 + \frac{1}{x}}{3 - \frac{2}{x}}$

Apply limit

$\Rightarrow \lim_{x \to \infty} \dfrac{2 + \cancel{\frac{1}{\infty}}^{0}}{3 - \underbrace{\cancel{\frac{2}{\infty}}}_{0}}$

$= \dfrac{2}{3}$ Ans

Evaluation of limits using some standard limits

Q. Evaluate

Ex - $\lim_{n \to \infty} \dfrac{1+2+3+\ldots+n}{n^2}$

$1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$

$\Rightarrow \lim_{n \to \infty} \dfrac{\frac{n(n+1)}{2}}{n^2}$

$\Rightarrow \lim_{n \to \infty} \dfrac{n^2+n}{2n^2} \quad \left(\dfrac{\infty}{\infty}\right)$

$\Rightarrow \dfrac{1}{2} \lim_{n \to \infty} \dfrac{\frac{\cancel{n^2}}{\cancel{n^2}} + \frac{\cancel{n}}{n^{\cancel{2}}}}{\frac{\cancel{n^2}}{\cancel{n^2}}}$

$\Rightarrow \dfrac{1}{2} \lim_{n \to \infty} 1 + \dfrac{1}{n} = \dfrac{1}{2}\left(1 + \cancel{\dfrac{1}{\infty}}_{0}\right) = \dfrac{1}{2}$

Ex- $\lim_{n\to\infty} \dfrac{1^2+2^2+3^2+\dots\dots+n^2}{n^3}$ | $1^2+2^2+3^2+\dots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$\Rightarrow \lim_{x\to\infty} \dfrac{\frac{n(n+1)(2n+1)}{6}}{n^3}$

$\Rightarrow \lim_{x\to\infty} \dfrac{\cancel{n}(n+1)(2n+1)}{6n^{\cancel{3}\,2}}$

$\Rightarrow \frac{1}{6}\lim_{x\to\infty} \dfrac{n\left(1+\frac{1}{n}\right)n\left(2+\frac{1}{n}\right)}{n^2}$

Apply limit

$\Rightarrow \frac{1}{6}\left(1+\cancelto{0}{\frac{1}{\infty}}\right)\left(2+\cancelto{0}{\frac{1}{\infty}}\right)$

$\Rightarrow \frac{2}{6}$

$\Rightarrow \frac{1}{3}$ Ans

Ex- $\lim_{n\to\infty} \dfrac{1^3+2^3+3^3+\dots+n^3}{n^4}$ | $1^3+2^3+3^3+\dots+n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

$\Rightarrow \lim_{n\to\infty} \dfrac{\left[\frac{n(n+1)}{2}\right]^2}{n^4}$

$\Rightarrow \lim_{x\to\infty} \dfrac{n^2(n+1)^2}{4n^{\cancel{4}\,2}}$ $\Rightarrow \frac{1}{4}\lim_{x\to\infty} \dfrac{n^2+2n+1}{n^2}$

$\Rightarrow \frac{1}{4}\lim_{n\to\infty}$ ~~$n^2\left(1^2+\frac{1}{n}\right)$~~ $\Rightarrow \frac{1}{4}\lim_{n\to\infty} \frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}$

$$= \frac{1}{4} \lim_{x \to \infty} \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$= \frac{1}{4}\left(1 + \underbrace{\cancel{\frac{2}{\infty}}}_{0} + \underbrace{\cancel{\frac{1}{\infty^2}}}_{0}\right)$$

$$= \frac{1}{4}$$ Ans

Prove the following limits

① $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, a > 0$

$a^2 - b^2 = (a-b)(a+b)$
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
$a^4 - b^4 = (a-b)(a^3 + a^2b + ab^2 + b^3)$

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + b^{n-1})$

Proof

LHS

$$\lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}(x^{n-1} + x^{n-2}a + \dots + a^{n-1})}{\cancel{(x-a)}}$$

$$= \lim_{x \to a} (x^{n-1} + x^{n-2} \cdot a + \dots + a^{n-1})$$

$$= a^{n-1} + a^{n-2} \cdot a + \dots + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + \dots + a^{n-1}$$

$$= na^{n-1}$$ RHS

ENGINEERING MATHEMATRCS

9/5/22

(2) $\lim_{x \to a} \frac{a^x - 1}{x} = \log_e a$

Let $a^x - 1 = y$

$\Rightarrow a^x = 1 + y$

$\Rightarrow \log a^x = \log(1+y)$

$\Rightarrow x \log a = \log(1+y)$

$\Rightarrow x = \frac{\log 1+y}{\log a}$

When $x \to 0,\ y \to 0$

$$\Rightarrow \lim_{y \to 0} \frac{y}{\frac{\log(1+y)}{\log a}} \quad \text{LHS}$$

$$\Rightarrow \lim_{y \to 0} \frac{\log a}{\frac{\log(1+y)}{y}}$$

$$\Rightarrow \lim_{y \to 0} \frac{\log a}{\frac{\log(1+y)}{y}} \qquad \left\| \lim_{y \to 0} \frac{\log(1+y)}{y} = 1 \right.$$

$$\Rightarrow \frac{\log a}{1}$$

$$\Rightarrow \frac{\log a}{\log e}$$

$$\Rightarrow \log_e a \quad \text{RHS}$$

(3) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Proof

LHS

$$\lim_{x \to 0} \frac{e^x - 1}{x} \qquad \left\| e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right.$$

$$\Rightarrow \lim_{x \to 0} \frac{\left(\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \cancel{1}}{x}$$

$$\Rightarrow \lim_{x \to 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x}$$

$$\Rightarrow \lim_{x \to 0} \frac{x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right)}{x}$$

$$\Rightarrow \lim_{x\to 0} \; 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\cdots$$

$$\Rightarrow 1 + \frac{\cancel{0}}{\cancel{2}!} + \frac{\cancel{0}}{\cancel{3}!} + \cdots\cdots$$

$\Rightarrow 1$ $\lfloor$ RHS

(4) $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

Proof

LHS

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}}$$

$$\Rightarrow \lim_{x\to 0} \; 1 + \frac{1}{x}\cdot x + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}\cdot x^2 + \frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}\cdot x^3 + \cdots$$

$$\Rightarrow \lim_{x\to 0} \; 1 + 1 + \frac{1-x}{2!} + \frac{(1-x)(1-2x)}{3!} + \cdots\cdots$$

Apply limit

$$\Rightarrow 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\cdots$$

$$\Rightarrow 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\cdots$$

$\Rightarrow e$ $\lfloor$ RHS

(5) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$

Proof

LHS

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$$

Let, $\frac{1}{x} = y$

$\Rightarrow x = \frac{1}{y}$

When $x = \infty$, $y \to \frac{1}{\infty} = 0$

$\Rightarrow \lim_{y \to 0} (1+y)^{\frac{1}{y}}$

$= e$ | RHS

(6) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

Proof

LHS

$\lim_{x \to 0} \frac{\log(1+x)}{x}$

Let, $\log(1+x) = y$

$\Rightarrow e^{\log(1+x)} = e^y$

$\Rightarrow 1 + x = e^y$

$\Rightarrow x = e^y - 1$

When $x \to 0$, $y \to \log 1 = 0$

$\Rightarrow \lim_{y \to 0} \frac{y}{e^y - 1}$

$\Rightarrow \lim_{y \to 0} \frac{1}{\frac{e^y - 1}{y}}$ || $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

$\Rightarrow 1$ | RHS

Evaluate

Q. $\lim_{x \to 0} \frac{e^{3x} - e^{x}}{x}$

soln - $\lim_{x \to 0} \frac{e^{x}(e^{2x} - 1)}{x}$ || $\lim_{x \to 0} \frac{e^{x} - 1}{x} = 1$

$= \lim_{x \to 0} e^{x} \lim_{x \to 0} \frac{e^{2x} - 1}{x}$

$= \lim_{x \to 0} e^{x} \lim_{x \to 0} \frac{2(e^{2x} - 1)}{2x}$

$= 2 \lim_{x \to 0} e^{x} \lim_{x \to 0} \frac{e^{2x} - 1}{2x}$

Apply limit

$= 2 \cdot e^{0} \cdot 1$

$= 2 \cdot 1 \cdot 1$

$= 2$ Ans

Q. $\lim_{x \to 0} \frac{3^{x} - 2^{x}}{4^{x} - 3^{x}}$

soln - $\lim_{x \to 0} \frac{\frac{3^{x} - 1 - 2^{x} + 1}{x}}{\frac{4^{x} - 1 - 3^{x} + 1}{x}}$ || $\lim_{x \to 0} \frac{a^{x} - 1}{x} = \log_e a$

$= \lim_{x \to 0} \frac{\frac{3^{x} - 1 - (2^{x} - 1)}{x}}{\frac{4^{x} - 1 - (3^{x} - 1)}{x}}$

$$= \frac{\lim\limits_{x\to 0}\left(\frac{3^x-1}{x}\right) - \lim\limits_{x\to 0}\left(\frac{2^x-1}{x}\right)}{\lim\limits_{x\to 0}\left(\frac{4^x-1}{x}\right) - \lim\limits_{x\to 0}\left(\frac{3^x-1}{x}\right)}$$

$$= \frac{\ln 3 - \ln 2}{\ln 4 - \ln 3}$$

$$= \frac{\ln 3/2}{\ln 4/3}$$

Q. $\lim\limits_{x\to 2} \dfrac{\log(3x-5)}{x-2}$

Soln - Let, $x-2 = u$

$\Rightarrow x = u+2$

$\| \lim\limits_{x\to 0} \dfrac{\log(1+x)}{x} = 1$

When $x \to 2 \Rightarrow u \to x-2$

$= 2-2$

$= 0$

$$\lim_{u\to 0} \frac{\log(3(u+2)-5)}{u}$$

$$\Rightarrow \lim_{u\to 0} \frac{\log(3u+6-5)}{u}$$

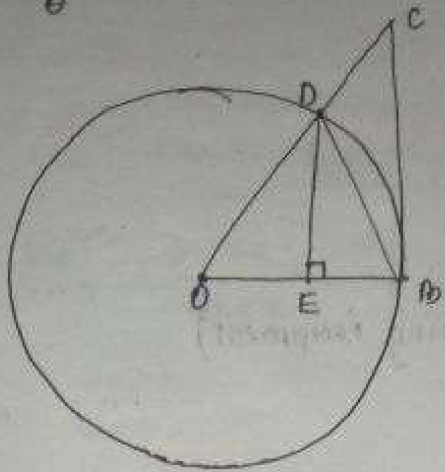
$$\Rightarrow \lim_{u\to 0} \frac{\log(1+3u)}{u}$$

$$\Rightarrow \lim_{x\to 0} \frac{3\log(1+3u)}{3u}$$

$$\Rightarrow 3\lim_{u\to 0} \frac{\log(1+3u)}{3u}$$

$\Rightarrow 3.1 = 3$ Ans

Q. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$



Proof

Take a unit circle
radius $r = 1$

BC is the tangent

Join BD, Draw $DE \perp OB$

Here, $\angle BOD = \theta = \angle BOC$

Here,

Area of $\triangle OBD$ < Area of arc OBD < Area of $\triangle OBC$

$\Rightarrow \frac{1}{2} \times OB \times DE < \frac{1}{2} r^2 \theta < \frac{1}{2} \times OB \times BC$ $\quad \Vert$ $r = 1$, $OB = 1$

$\Rightarrow \frac{1}{2} \times 1 \times DE < \frac{1}{2} \times 1 \times \theta < \frac{1}{2} \times 1 \times BC$

$\Rightarrow DE < \theta < BC$ —— (1)

In $\triangle OED$,

$\sin\theta = \frac{DE}{OD}$

$= \frac{DE}{1}$

$\Rightarrow DE = \sin\theta$ —— (2)

In $\triangle OBC$,

$\tan\theta = \frac{BC}{OB} = \frac{BC}{1}$

$\Rightarrow BC = \tan\theta$ —— (3)

Substituting the values of DE and BC in eqn ⑤

$\sin\theta < \theta < \tan\theta$

$\Rightarrow \dfrac{\sin\theta}{\sin\theta} < \dfrac{\theta}{\sin\theta} < \dfrac{\tan\theta}{\sin\theta}$

$\Rightarrow 1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$

(Taking reciprocal)

$\Rightarrow 1 > \dfrac{\sin\theta}{\theta} > \cos\theta$

Now,

$\lim\limits_{\theta\to 0} 1 > \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} > \lim\limits_{\theta\to 0} \cos\theta$

$\Rightarrow 1 > \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} > 1$

By Sandwich theorem

$\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ Proved

Sandwich theorem

If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = l$ and a function ϕ is such that $f(x) \le \phi(x) \le g(x)$ for all x in a deleted neighbourhood of a then $\lim\limits_{x\to a} \phi(x) = l$

(8) $\lim\limits_{\theta\to 0} \dfrac{\tan\theta}{\theta} = 1$

LHS

$\lim\limits_{\theta\to 0} \dfrac{\tan\theta}{\theta}$

$= \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta \cdot \cos\theta}$

$= \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} \lim\limits_{\theta\to 0} \dfrac{1}{\cos\theta}$

$= 1 \cdot \frac{1}{\cos 0}$

$= 1$

ENGINEERING MATHEMATICS

12/5/22

Ex - find $\lim_{x \to 0} \frac{\sin 2x}{\sin 3x}$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

soln - $\lim_{x \to 0} \frac{\sin 2x}{\sin 3x}$

$= \lim_{x \to 0} \frac{2 \frac{\sin 2x}{2x}}{3 \frac{\sin 3x}{3x}}$

$= \frac{2}{3} \frac{\lim_{x \to 0} \frac{\sin 2x}{2x}}{\lim_{x \to 0} \frac{\sin 3x}{3x}}$

$= \frac{2}{3} \cdot \frac{1}{1} = \frac{2}{3}$ Ans

Q. $\lim_{x \to \pi} \frac{\sin x}{\pi - x}$

Let $\pi - x = u$

where $\Rightarrow x = \pi - u$

$x \to \pi, u \to \pi - \pi = 0$

$= \lim_{x \to 0} \frac{\sin(\pi - u)}{u}$

$= \lim_{x \to 0} \frac{\sin u}{u}$

$= 1$ Ans

Continuity

* Continous fuction

A function 'f' is said to be continous at a point a ∈ D_f if

i) $f(x)$ has definite value $f(a)$ at $x = a$

ii) $\lim_{x \to a} f(x)$ exists

iii) $\lim_{x \to a} f(x) = f(a)$

(i.e limiting value is equal to functional value at that point)

→ If one or more of the above conditions fail, the function 'f' is said to be discontinous at $x = a$.

Ex - Examine the continuity of the function

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

at $x = 0$

soln -

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

i) $f(0) = 0$

Now, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x}{|x|}$

LHL

$$\lim_{x \to 0^-} \frac{x}{|x|}$$

$$= \lim_{h \to 0} \frac{0-h}{|0-h|}$$

$$= \lim_{h \to 0} \frac{-h}{h}$$

$= -1$

RHL

$$\lim_{x\to 0^+} \frac{x}{|x|}$$

$$= \lim_{h\to 0} \frac{0+h}{|0+h|}$$

$$= \lim_{h\to 0} \frac{h}{h}$$

$$= 1$$

$\therefore$ LHL $\neq$ RHL

$\lim_{x\to 0} f(x)$ does not exist.

The function is discontinous.

Q. Test for continuity

$$f(x) = \begin{cases} \frac{\sin 2x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

at $x = 0$

soln - Here $f(0) = 1$

Now, $\lim\limits_{x \to 0} f(x)$

$$= \lim_{x \to 0} \frac{\sin 2x}{x}$$

$$= \lim_{x \to 0} 2 \frac{\sin 2x}{2x}$$

$$= 2 \lim_{x \to 0} \frac{\sin 2x}{2x}$$

$$= 2 \cdot 1$$

$$= 2$$

~~So~~ since, $\lim\limits_{x \to 0} f(x) \neq f(0)$

So, $f(x)$ is discontinous at $x = 0$

Q. Test for continuity

$$f(x) = \begin{cases} \frac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

at $x = 0$

soln - Here, $f(0) = 2$

Now, $\lim_{x \to 0} f(x)$

$= \lim_{x \to 0} \frac{\sin 2x}{x}$

$= \lim_{x \to 0} \frac{2 \sin 2x}{2x}$

$= 2 \lim_{x \to 0} \frac{\sin 2x}{2x}$

$= 2 \cdot 1$

$= 2$

since, $\lim_{x \to 0} f(x) = f(0)$

so, $f(x)$ is a continous at $x = 0$

Q. Test for continuity

$$f(x) = \begin{cases} \frac{x^2 - a^2}{x - a}, & x \neq a \\ a, & x = a \end{cases}$$

at $x = a$

soln - Here $f(a) = a$

$\lim_{x \to a} f(x)$

$$= \lim_{x \to a} \frac{x^2 - a^2}{x - a}$$

$$= \lim_{x \to a} \frac{(x-a)(x+a)}{(x-a)}$$

$$= \lim_{x \to a} (x+a)$$

Applying limit

$= a + a$

$= 2a$

Since, $\lim_{x \to a} f(x) \neq f(a)$

The $f(x)$ is discontinous at $x = a$

Q. Test for continuity

$$f(x) = \begin{cases} (1+2x)^{\frac{1}{x}}, & \text{if } x \neq 0 \\ e^2, & \text{if } x = 0 \end{cases}$$

at $x = 0$

Soln - Here, $f(0) = e^2$

Now, $\lim_{x \to 0} f(x)$

$$= \lim_{x \to 0} (1+2x)^{\frac{1}{x}}$$

Let $2x = u$

$\Rightarrow x = \frac{u}{2}$

When $x \to 0$

$\Rightarrow u \to 0$

So, $\lim_{u \to 0} (1+u)^{\frac{1}{\frac{u}{2}}}$

$= \lim_{u \to 0} (1+u)^{\frac{2}{u}}$

$= \lim_{u \to 0} \left((1+u)^{\frac{1}{u}}\right)^2$

$= \left(\lim_{u \to 0} (1+u)^{\frac{1}{u}}\right)^2$

$= e^2$

Since $\lim_{x \to 0} f(x) = f(0)$

f is continous at $x = 0$

Q. If

$$f(x) = \begin{cases} ax^2 + b, & \text{if } x < 1 \\ 1, & \text{if } x = 1 \\ 2ax - b, & \text{if } x > 1 \end{cases}$$

is continous at $x = 1$, then find a and b.

soln- Given that

$f(x)$ is continous at $x = 1$

Also, $f(1) = 1$

i.e $\lim_{x \to 1} f(x) = 1 = f(1)$.

Now, $\lim_{x \to 1} f(x) = 1$

LHL

$\lim_{x \to 1^-} f(x)$

$= \lim_{x \to 1^-} ax^2 + b$

$= \lim_{h \to 0} a(1-h)^2 + b$

$= a(1-0)^2 + b$

$= a+b$

RHL

$\lim_{x \to 1^+} f(x)$ &

$= \lim_{x \to 1^+} 2ax - b$

$= \lim_{x \to 1^+} 2a(1+h) - b$

$= 2a(1+0) - b$

$= 2a - b$

Since,

$LHL = RHL = f(1) = 1$

$\Rightarrow a+b = 1$ ——— (1)

$2a - b = 1$ ——— (2)

$a+b = 1$

$\Rightarrow a = 1-b$

Substituting in equation (2)

$2a - b = 1$

$\Rightarrow 2(1-b) - b = 1$

$\Rightarrow 2 - 2b - b = 1$

$\Rightarrow -3b = 1-2 = -1$

$\Rightarrow b = \frac{-1}{-3} = \frac{1}{3}$

$\therefore a = 1 - b = 1 - \frac{1}{3} = \frac{2}{3}$ Ans

Differentiation

Let $y = f(x)$

x → independent variable

y → dependent variable

Let δx and δy be small increment in x and y respectively.

i.e $y + \delta y = f(x + \delta x)$

$\Rightarrow \delta y = f(x+\delta x) - y$

$= f(x+\delta x) - f(x)$.

Average change in y

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

The instantenous rate of change of y at the value of x is given by

$$\boxed{\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x)}{\delta x}} \quad \text{——} \quad (*)$$

If this limit $(*)$ exists, it is called derivative of y with respect to x.

so, $(*)$ can be written as

$$\boxed{\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}}$$

Notation of derivatives

f', $\frac{dy}{dx}$, Dy, Df

$\frac{dy}{dx}$ → Derivative of y withrespect to x.

* The process of finding the derivative of a function is known as differentiation.

* Let $c \in (a, b)$ and f be a function.

$$\left.\frac{dy}{dx}\right]_{x=c} = f'(c)$$

$$= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \quad \text{——— (1)}$$

* In (1) if the limit exists when $h \to 0^+$, the limit is called the right hand derivative of f at 'c' and is denoted by $f'(c^+)$.

* Similarly if the limit exists when $h \to 0^-$ is called the left hand derivative of f at 'c' and is denoted by $f'(c^-)$.

$$\rightarrow f'(c^-) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{h} \quad h > 0$$

$$f'(c^+) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \quad h > 0$$

if $f'(c^-) = f'(c^+)$

Then we say the function is differentiable at 'c'.

Derivative of some standard functions

(1) $$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

$$y = x^n$$

Let δx and δy be small increment in x and y respectively.

$$y + \delta y = (x + \delta x)^n$$

$$\Rightarrow \delta y = (x + \delta x)^n - y$$

$$= (x + \delta x)^n - x^n$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{(x+\delta x)^n - x^n}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{\delta x}$$

$$\left[\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1} \right.$$

$$= \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{x+\delta x - x}$$

* Let $x + \delta x = z$
When $\delta x \to 0$
$z \to x$

$$= \lim_{z \to x} \frac{z^n - x^n}{z - x}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$ Proved

* $\frac{d}{dx}(x^2) = 2x^{2-1}$
$= 2x$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{d}{dx}(x^5) = 5x^4$$

$$\frac{d}{dx}(x^7) = 7x^6$$

$$\frac{d}{dx}(x^{10}) = 10x^9$$

$$\frac{d}{dx}(x) = 1$$

(2) $\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$

$y = a^x$

Let δx and δy be small increment in x and y respectively.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{a^{x+\delta x} - a^x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{a^x (a^{\delta x} - 1)}{\delta x}$$

$$= a^x \lim_{\delta x \to 0} \left(\frac{a^{\delta x} - 1}{\delta x} \right)$$

$$= a^x \ln a \qquad \left|\right| \quad \lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$ Proved

(3) $\frac{d}{dx}(e^x) = e^x$

$y = e^x$

Let δx and δy be the small increment in x and y respectively

$y + \delta y = e^{x+\delta x}$

$\Rightarrow \delta y = e^{x+\delta x} - y$

$= e^{x+\delta x} - e^x$

$\Rightarrow \frac{\delta y}{\delta x} = \frac{e^{x+\delta x} - e^x}{\delta x}$

$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{dy}{dx}$

$= \lim_{\delta x \to 0} \frac{e^{x+\delta x} - e^x}{\delta x}$

$= \lim_{\delta x \to 0} \frac{e^x(e^{\delta x} - 1)}{\delta x}$

$= e^x \cdot \lim_{\delta x \to 0} \frac{e^{\delta x} - 1}{\delta x}$ $\quad || \lim_{x \to 0} \frac{e^x - 1}{x} = 1$

$= e^x \cdot 1$

$= e^x$ Proved

(4) $\frac{d}{dx}(\sin x) = \cos x$

$y = \sin x$

Let δx and δy be the small increment in x and y respectively.

$y + \delta y = \sin(x + \delta x)$

$\Rightarrow \delta y = \sin(x + \delta x) - y$

$= \sin(x + \delta x) - \sin x \Rightarrow \frac{\delta y}{\delta x} = \frac{\sin(x+\delta x) - \sin x}{\delta x}$

$$\frac{d}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x}$$

|| $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(\frac{x+\delta x+x}{2}\right) \cdot \sin\left(\frac{x+\delta x - x}{2}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(\frac{2x+\delta x}{2}\right) \cdot \sin\frac{\delta x}{2}}{\delta x}$$

$$= \lim_{\delta x \to 0} \cos\left(\frac{2x+\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

|| $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Apply limit

$$= \cos\left(\frac{2x+0}{2}\right) \cdot 1$$

$$= \cos\frac{2x}{2} = \cos x$$

(5) $\frac{d}{dx}(\cos x) = -\sin x$

$y = \cos x$

Let δx and δy be the small increment in x and y respectively.

$y + \delta y = \cos(x + \delta x)$

$\Rightarrow \delta y = \cos(x + \delta y) - y$

$= \cos(x + \delta y) - \cos x$

$\Rightarrow \frac{\delta y}{\delta x} = \frac{\cos(x+\delta y) - \cos x}{\delta x}$

$$\frac{d}{dx} = \lim_{\delta x \to 0} \frac{dy}{dx}$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cos(x+\delta x) - \cos x}{\delta x}$$ // $\cos C - \cos D = -2\sin \frac{C+D}{2} \cdot \sin \frac{C-D}{2}$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(\frac{x+\delta x+x}{2}\right) \cdot \sin\left(\frac{\cancel{x}+\delta x-\cancel{x}}{2}\right)}{\delta x}$$

$$= -\lim_{\delta x \to 0} \frac{2\sin\left(\frac{2x+\delta x}{2}\right) \cdot \sin \frac{\delta x}{2}}{\delta x}$$

$$= -\lim_{\delta x \to 0} \sin\left(\frac{2x+\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$= -\sin\left(\frac{2x+0}{2}\right)$$

$$= -\sin \frac{2x}{2}$$

$$= -\sin x$$

(6) $\frac{d}{dx}(\tan x) = \sec^2 x$

$y = \tan x$

Let δx and δy be the small increment in x and y respectively.

$$y + \delta y = \tan(x+\delta x)$$

$$\Rightarrow \delta y = \tan(x+\delta x) - y$$

$$= \tan(x+\delta x) - \tan x$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\tan(x+\delta x) - \tan x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(\cancel{x}+\delta x-\cancel{x})}{\cos(x+\delta x)\cos x \cdot \delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \cdot \frac{1}{\lim_{\delta x \to 0} \cos(x+\delta x) \cdot \cos x}$$

$$= 1 \cdot \frac{1}{\cos x \cdot \cos x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \quad \text{Proved}$$

(7) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(8) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

(9) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

(10) $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

(11) $\frac{d}{dx}(\ln x) = \frac{1}{x}$

(12) $\frac{d}{dx}(c) = 0$, $c =$ Constant

(13) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(14) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(15) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

(16) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

(17) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$

(18) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Algebra of Derivatives

Let u and v be two differentiable functions of x. Then

1) $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

2) $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$

3) $\frac{d}{dx}(uv) = v.\frac{du}{dx} + u.\frac{dv}{dx}$

4) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$, $v \neq 0$

*5) $\frac{d}{dx}(cu) = c.\frac{du}{dx}$

Ex - $y = x^7 + x^2$

find $\frac{dy}{dx} = ?$

Soln - $y = x^7 + x^2$

$\frac{dy}{dx} = \frac{d}{dx}(x^7 + x^2)$

$= \frac{d}{dx}(x^7) + \frac{d}{dx}(x^2)$

$= 7x^6 + 2x$

Q. $y = \sin x - \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x - \cos x)$$

$$= \frac{d}{dx}(\sin x) - \frac{d}{dx}(\cos x)$$

$$= \cos x - (-\sin x)$$

$$= \cos x + \sin x$$

Q. $y = \sin x \cdot \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x \cdot \cos x)$$

$$= \cos x \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(\cos x)$$

$$= \cos x \cdot \cos x + \sin x \cdot (-\sin x)$$

$$= \cos x^2 - \sin x^2$$

$$= \cos 2x$$

Q. $\frac{d}{dx}\left(\frac{x}{\sin x}\right)$

$$= \frac{\sin x \cdot \frac{d}{dx}(x) - x \cdot \frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$= \frac{\sin x \cdot 1 - x \cdot \cos x}{\sin^2 x}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x}$$ Ans

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Q. $y = \dfrac{1 - \cos x}{1 + \cos x}$

find $\dfrac{dy}{dx} = ?$

soln - $\dfrac{dy}{dx} = \dfrac{(1+\cos x)\dfrac{d}{dx}(1-\cos x) - (1-\cos x)\dfrac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$

$$= \frac{(1+\cos x)(0+\sin x) - (1-\cos x)(0-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\sin x\,(1+\cos x) + \sin(1-\cos x)}{(1+\cos x)^2}$$

$$= \frac{\sin x\,\{1+\cancel{\cos x}+1-\cancel{\cos x}\}}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2}$$ Ans

Q. $y = \dfrac{1-\tan x}{1+\tan x}$

find $\dfrac{dy}{dx} = ?$

soln -

$$\frac{dy}{dx} = \frac{(1+\tan x)\frac{d}{dx}(1-\tan x) - (1-\tan x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)(0-\sec^2 x) - (1-\tan x)(1+\sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec^2 x\,\{-1-\tan x - 1 + \tan x\}}{(1+\tan x)^2}$$

$$= \frac{-2\sec^2 x}{(1+\tan x)^2} \quad \text{Ans}$$

Q. $y = \dfrac{x^2-1}{x^3+1}$

find $\dfrac{dy}{dx} = ?$

Ans - $$\frac{dy}{dx} = \frac{(x^3+1)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x^3+1)}{(x^3+1)^2}$$

$$= \frac{(x^3+1)(2x-0) - (x^2-1)(3x^2+0)}{(x^3+1)^2}$$

$$= \frac{(x^3+1)(2x) - (x^2-1)(3x^2)}{(x^3+1)^2}$$

$$= \frac{2x(x^3+1) - 3x^2(x^2-1)}{(x^3+1)^2} \quad \text{Ans}$$

Derivative of a composite function (The chain Rule)

Let, $y = f(u)$ be a differentiable function of u

$u = g(x)$ be a differentiable function of x

Then, $y = f \circ g$ is a composite function of x.

Then, $\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$

Example

$y = \cos x^2$

Soln —

Let $u = x^2$

$y = \cos u$

$\Rightarrow \frac{dy}{du} = -\sin u$

$u = x^2$

$\frac{du}{dx} = 2x$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= -\sin u \cdot 2x$

$= -\sin x^2 \cdot 2x$

$= -2x \sin x^2$ Ans

Example

$y = \cos x^2$

Soln —

$\frac{dy}{dx} = -\sin x^2 \frac{d}{dx}(x^2)$

$= -\sin x^2 \cdot 2x = -2x \sin x^2$ Ans

Example

$$y = (x^2+2x-1)^5$$

Soln -

$$\frac{dy}{dx} = 5(x^2+2x-1)^4 \cdot \frac{d}{dx}(x^2+2x-1)$$

$$= 5 \cdot (x^2+2x-1)^4 (2x+2)$$

$$= 5 \cdot (x^2+2x-1) \, 2(x+1)$$

$$= 10(x+1)(x^2+2x-1)^4$$

Q. $y = \sec(\tan x)$

Soln -

$$\frac{dy}{\cancel{\tan}\, dx} = \sec(\tan x) \cdot \tan(\tan x) \frac{d}{dx}(\tan x)$$

$$= \sec(\tan x) \cdot \tan(\tan x) \cdot \sec^2 x$$ Ans

Q. $y = \frac{e^{3x^2}}{\ln \sin x}$, find $\frac{dy}{dx} = ?$

Soln -

$$y = \frac{e^{3x^2}}{\ln \sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\ln \sin x) \cdot \frac{d}{dx} - e^{3x^2} \cdot \frac{d}{dx}(\ln \sin x)}{(\ln \sin x)^2}$$

$$= \frac{\ln(\sin x)\, e^{3x^2} \frac{d}{dx}(3x^2) - e^{3x^2} \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x)}{(\ln \sin x)^2}$$

$$= \frac{\ln(\sin)\, e^{3x^2} \cdot 6x - e^{3x^2} \frac{1}{\sin x} \cdot \cos x}{(\ln \sin x)^2}$$

$$= \frac{e^{3x^2}(6x \ln \sin x - \cot x)}{(\ln \sin x)^2}$$ Ans

Assignment
Elements (XII)
Ex - 7(c)

Methods of differentiation

1) Differentiation using Logarithm

→ When a function appears as an exponent of another function we make use of Logarithm.

Ex - Differentiate $(\sin x)^{\tan x}$

Soln - Let $y = (\sin x)^{\tan x}$

Taking logarithm to both sides

$$\ln y = \ln(\sin x)^{\tan x} \qquad \| \log x^m = m \log x$$

$$\Rightarrow \ln y = \tan x \cdot \ln(\sin x)$$

Now, differentiating both sides with respect to x. $\| \frac{d}{dx}(uv) = v \cdot \frac{du}{dx} + u \cdot \frac{dv}{dx}$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(\sin x) \cdot \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\ln(\sin x))$$

$$= \ln(\sin x) \cdot \sec^2 x + \tan x \frac{1}{\sin x} \cdot \cos x$$

$$= \sec^2 x \cdot \ln(\sin x) + \tan x \cdot \cot x$$

$$\Rightarrow \frac{dy}{dx} = y\left[\sec^2 x \cdot \ln(\sin x) + 1\right]$$

$$= (\sin x)^{\tan x}\left[\sec^2 x \cdot \ln(\sin x) + 1\right]$$

Ex- Differentiate $(\log x)^{\tan x}$

Soln- Let $y = (\log x)^{\tan x}$

Taking logarithm to both sides.

$$\ln y = \ln(\log x)^{\tan x}$$

$$= \tan x \cdot \ln(\log x)$$

Now, differentiating both sides with respect to x.

$$\frac{1}{y} \frac{dy}{dx} = \ln(\log x) \cdot \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\ln(\log x))$$

$$= \ln(\log x) \cdot \sec^2 x + \tan x \frac{1}{\log x} \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\ln \log x \cdot \sec^2 x + \frac{\tan x}{x \log x}\right]$$

$$= (\log x)^{\tan x}\left[\ln(\log x) \cdot \sec^2 x + \tan x \frac{1}{\log x} \cdot \frac{1}{x}\right]$$

→ When a given function is expressed as a product of several functions, we use logarithmic differentiation.

Ex- Differentiate

$$y = \frac{(x-1)^2\sqrt{3x^2-1}}{x^7(6-7x^2)^{\frac{3}{2}}}$$

soln - Taking logarithm to both sides

$$\ln y = \ln\left[\frac{(x-1)^2\sqrt{3x^2-1}}{x^7(6-7x^2)^{\frac{3}{2}}}\right]$$

$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

$\ln(xy) = \ln x + \ln y$

$$= \ln\left\{(x-1)^2\sqrt{3x^2-1}\right\} - \ln\left\{x^7(6-7x^2)^{\frac{3}{2}}\right\}$$

$$= \ln(x-1)^2 + \ln(3x^2-1)^{\frac{1}{2}} - \left\{\ln x^7 + \ln(6-7x^2)^{\frac{3}{2}}\right\}$$

$$= 2\ln(x-1) + \frac{1}{2}\ln(3x^2-1) - 7\ln x + \frac{3}{2}\ln(6-7x^2)$$

Now differentiating both sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = 2\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{3x^2-1}\cdot\frac{d}{dx}(3x^2-1) - 7\frac{1}{x} - \frac{3}{2}\cdot\frac{1}{6-7x^2}\cdot\frac{d}{dx}(6-7x^2)$$

$$= \frac{2}{x-1} + \frac{1}{\cancel{2}(3x^2-1)}\cdot\overset{3}{\cancel{6}}x - \frac{7}{x} - \frac{3}{\cancel{2}(6-7x^2)}\left(-\overset{7}{\cancel{14}}x\right)$$

$$= \frac{2}{x-1} + \frac{3x}{3x^2-1} - \frac{7}{x} + \frac{21}{6-7x^2}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x-1} + \frac{3x}{3x^2-1} - \frac{7}{x} + \frac{21}{6-7x^2}\right]$$

$$= \frac{(x-1)^2\sqrt{3x^2-1}}{x^7(6-7x^2)^{\frac{3}{2}}}\left[\frac{2}{x-1}+\frac{3x}{3x^2-1}-\frac{7}{x}+\frac{21}{6-7x^2}\right]$$

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ENGINEERING MATHEMATICS

Differentiation of implict function

Implict function

If y can't be written interms of x only uniquely then y is called an implict function.

$$xy^2+x^2y=5$$
$$y[xy+x^2]$$

Ex- find $\frac{dy}{dx}$ if $x^2+y^2-a^2=0$

Soln- $x^2+y^2-a^2=0$

Differentiation both sides with respect to x

$$2x+2y\frac{dy}{dx}-0=0$$

$$\Rightarrow \cancel{2}y\frac{dy}{dx}=-\cancel{2}x$$

$$\Rightarrow \frac{dy}{dx}=\frac{-x}{y}$$ Ans

Ex- find $\frac{dy}{dx}$ if $y^3-3x^2y-2x=10$

Soln- $y^3-3x^2y-2x=10$

Differentiating both sides with respect to x

$$3y^2.\frac{dy}{dx}-3\left(y.2x+x^2.\frac{dy}{dx}\right)-2=0$$

$$\Rightarrow 3y^2.\frac{dy}{dx}-6xy-3x^2.\frac{dy}{dx}-2=0$$

$$\Rightarrow \frac{dy}{dx}(3y^2-3x^2)=2+6xy$$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2+6xy}{3y^2-3x^2}$

$= \dfrac{2(1+3xy)}{3(y^2-x^2)}$ Ans

Ex- find $\dfrac{dy}{dx}$ if $xy^2 + x^2y + 1 = 0$

Soln- $xy^2 + x^2y + 1 = 0$

Differentiating both sides with respect to x

$y^2 \cdot 1 + x \cdot 2y \cdot \dfrac{dy}{dx} + y \cdot 2x + x^2 \dfrac{dy}{dx} + 0 = 0$

$\Rightarrow \dfrac{dy}{dx}(2xy + x^2) = -(y^2 + 2xy)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{(-y^2+2xy)}{(2xy+x^2)}$ Ans

Ex- find $\dfrac{dy}{dx}$ if $e^{xy} + y\sin x = 1$, $\dfrac{dy}{dx}$?

Soln- $e^{xy} + y\sin x = 1$

Differentiating both sides with respect to x

$e^{xy} \cdot \dfrac{d}{dx}(xy) + \sin x \cdot \dfrac{dy}{dx} + y\cos x = 0$

$\Rightarrow e^{xy} \cdot \left(y \cdot 1 + x \cdot \dfrac{dy}{dx}\right) + \sin x \dfrac{dy}{dx} + y\cos x = 0$

$\Rightarrow ye^{xy} + xe^{xy}\dfrac{dy}{dx} + \sin x \dfrac{dy}{dx} + y\cos x = 0$

$\Rightarrow \dfrac{dy}{dx}(xe^{xy} + \sin x) = -(ye^{xy} + y\cos x)$

$\Rightarrow \dfrac{dy}{dx} = \left(\dfrac{-ye^{xy} + y\cos x}{xe^{xy} + \sin x}\right)$

ENGINEERING MATHEMATICS

1/6/22

Q. $y^x = x^{\sin y}$, find $\frac{dy}{dx} = ?$

Soln- $y^x = x^{\sin y}$

$\ln(y^x) = \ln(x^{\sin y})$

$\Rightarrow x \ln y = \sin y \cdot \ln x$

Now differentiating both sides with respect to x

$\Rightarrow \ln y \cdot 1 + x \cdot \frac{1}{y} \frac{dy}{dx} = \ln x \cos y \frac{dy}{dx} + \sin y \cdot \frac{1}{x}$

$\Rightarrow \frac{dy}{dx}\left[\frac{x}{y} - \ln x \cos y\right] = \frac{\sin y}{x} - \ln y$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{\sin y}{x} - \ln y}{\frac{x}{y} - \ln x \cdot \cos y}$$

$$= \frac{\dfrac{\sin y - x \ln y}{x}}{\dfrac{x - y \ln x \cdot \cos y}{y}}$$

$$= \frac{y(\sin y - x \ln y)}{x(x - y \ln x \cdot \cos y)}$$ Ans

Differentiation of parametric function

Let, $x = \phi(t)$

$y = \psi(t)$, t is a parameter

Then,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \boxed{\frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{\psi'(t)}{\phi'(t)}}$$

Ex. find $\frac{dy}{dx}$ if $x = a(\cos t + t \sin t)$
$y = a(\sin t - t \cos t)$

soln - $y = a(\sin t - t \cos t)$

$$\frac{dy}{dt} = a\{\cos t - (\cos t \cdot 1 + t \cdot (-\sin t))\}$$

$$= a(\cancel{\cos t} - \cancel{\cos t} + t \sin t)$$

$$= at \sin t$$

$x = a(\cos t + t \sin t)$

$$\frac{dx}{dt} = \cancel{a[-\sin t + \sin t + t\cos t]}\ a[-\sin t + \sin t \cdot 1 + t \cdot \cos t]$$

$$= a(-\cancel{\sin t} + \cancel{\sin t} + t\cos t)$$

$$= at\cos t$$

$\therefore \frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{a t \sin t}{a t \cos t} = \tan t$

Ex. $x = 3\cos t - 2\cos^3 t$

$y = 3\sin t - 2\sin^3 t$

$\frac{dy}{dx} = ?$

soln - $y = 3\sin t - 2\sin^3 t$

$\frac{dy}{dt} = 3.\cos t - 2.3\sin^2 t \cdot \cos t$

$= 3\cos t\,(1 - 2\sin^2 t)$

$= 3\cos t \,.\, \cos 2t \quad // \; 1 - 2\sin^2 t = \cos 2t$

$x = 3\cos t - 2\cos^3 t$

$\frac{dx}{dt} = -3.\sin t - 2.3\cos^2 t \cdot (-\sin t)$

$= -3\sin t + 2.3\cos^2 t \,.\, \sin t$

$= 3\sin t\,(-1 + 2\cos^2 t)$

$= 3\sin t\,(2\cos^2 t - 1)$

$= 3\sin t \,.\, \cos 2t$

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$

$= \frac{\cancel{3}\cos t \,.\, \cancel{\cos 2t}}{\cancel{3}\sin t \,.\, \cancel{\cos 2t}}$

$= \cot t$

Differentiation with respect to a function

Let $y = f(x)$

$z = g(x)$, be two differentiable functions

Now, $\boxed{\frac{dy}{dz} = \frac{dy}{dx} / \frac{dz}{dx} = \frac{f'(x)}{g'(x)}}$

Ex- Differentiate $\tan^{-1}x$ with respect to $\cos^{-1}x$.

Soln - Let $y = \tan^{-1}x$

$z = \cos^{-1}x$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\frac{dz}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}}$$

$$= \frac{\frac{1}{1+x^2}}{\frac{-1}{\sqrt{1-x^2}}} = \frac{-\sqrt{1-x^2}}{1+x^2}$$

Ex- Differentiate $\sin x$ with respect to $\cot x$

Soln - Let $y = \sin x$

$z = \cot x$

$$\frac{dy}{dx} = \cos x$$

$$\frac{dz}{dx} = -\operatorname{cosec}^2 x$$

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\cos x}{-\operatorname{cosec}^2 x} = \frac{-\cos x}{\operatorname{cosec}^2 x}$$ Ans

ENGINEERING MATHEMATICS 4/6/22

Ex - Find y_2 if $y = x^5 + 4x^3 - 2x^2 + 1$

Soln - $y = x^5 + 4x^3 - 2x^2 + 1$

$$\frac{dy}{dx} = y_1 = 5x^4 + 12x^2 - 4x$$

$$y_2 = \frac{d}{dx}(5x^4 + 12x^2 - 4x)$$

$$= 20x^3 + 24x - 4$$

Ans

Ex - Find y_2 if $y = (ax+b)^m$

soln - $y = (ax+b)^m$

$\Rightarrow y_1 = m(ax+b)^{m-1}$

$= \frac{d}{dx}(ax+b)^{m-1} \cdot a$

$= am\,(ax+b)^{m-1}$

$\Rightarrow y_2 = am(m-1)(ax+b)^{m-2} \cdot a$

$= a^2 m(m-1)(ax+b)^{m-2}$ Ans

Ex - If $x = \sin t$, $y = \sin(pt)$ then show that

$$(1-x)^3 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$$

soln -

Proof $y = \sin(pt)$, $x = \sin t$

$\Rightarrow t = \sin^{-1} x$

$= \sin(p \sin^{-1} x)$

Now, differentiating both sides with respect to x.

$\frac{dy}{dx} = \cos(p\sin^{-1}x)\frac{d}{dx}(p\sin^{-1}x)$

$= \cos(p\sin^{-1}x) \cdot p \cdot \frac{1}{\sqrt{1-x^2}}$

$= \frac{1}{\sqrt{1-x^2}} \cdot \frac{dy}{dx} = p\cos(p\sin^{-1}x)$

Squaring both sides

$(1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2\cos^2(p\sin^{-1}x)$

$= p^2[1-\sin^2(p\sin^{-1}x)]$

$= p^2 - p^2\sin^2(p\sin^{-1}x)$

$(1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2 - p^2y^2 \quad -(1)$

Again differentiating eqn (1) with respect to x.

$\left(\frac{dy}{dx}\right)^2 \cdot (-2x) + (1-x^2)\, 2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2}$

$= 0 - p^2 \cdot 2y\frac{dy}{dx}$

$\Rightarrow -2x\left(\frac{dy}{dx}\right)^2 + 2(1-x^2)\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + 2p^2y\cdot\frac{dy}{dx} = 0$

Dividing by $2\frac{dy}{dx}$

$\Rightarrow -x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} + p^2y = 0$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\cdot\frac{dy}{dx} + p^2y = 0$ | Proved

Q. If $y = \tan^{-1}x$
Prove that
$(1+x^2)y_2 + 2xy_1 = 0$

Soln -
Proof

$y = \tan^{-1}x$

$\Rightarrow y_1 = \frac{1}{1+x^2}$

$\Rightarrow (1+x^2)y_1 = 1$ —— (1)

Differentiating eqn (1) with respect to x.

$y_1 2x + (1+x^2)y_2 = 0$

$\Rightarrow (1+x^2)y_2 + 2xy_1 = 0$ | Proved

Q. If $2y = x\left(1+\frac{dy}{dx}\right)$ Then show that y_2 is a constant.

Soln -
Proof

To show y_2 is a constant we have to prove that $\frac{d}{dx}(y_2) = 0$

i.e $y_3 = 0$

$2y = x\left(1+\frac{dy}{dx}\right)$

Differentiating both sides with respect to x.

$2\cdot\frac{dy}{dx} = \left(1+\frac{dy}{dx}\right)\cdot 1 + x\cdot\frac{d^2y}{dx^2}$

$\Rightarrow 2\frac{dy}{dx} = 1 + \frac{dy}{dx} + x\frac{d^2y}{dx^2}$

$\Rightarrow 2\frac{dy}{dx} - \frac{dy}{dx} - x\frac{d^2y}{dx^2} = 1$

$$\Rightarrow \frac{dy}{dx} - x\frac{d^2y}{dx^2} = 1$$

$$\Rightarrow y_1 - xy_2 = 1 \quad \text{--- (1)}$$

Again differentiating eqn (1) with respect to x,

$$y_2 - y_2 \cdot 1 - xy_3 = 0$$

$$\Rightarrow \cancel{y_2} - \cancel{y_2} - xy_3 = 0$$

$$\Rightarrow y_3 = 0$$

$\Rightarrow y_2$ is a constant | Proved

Partial differentiation

functions of two variables

Let $f : X \times Y \to Z$ is a function of two variables if there exists a unique element $z = f(x,y)$ in Z corresponding to every pair (x,y) in $X \times Y$.

$$z = f(x,y)$$

x, y → Independent variable

z → Dependent variable.

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

Notations

$$z = f(x,y)$$

$$\frac{\partial z}{\partial x} \to f_x \qquad \frac{\partial z}{\partial y} \to f_y \qquad \frac{\partial^2 z}{\partial x^2} \to f_{xx} \qquad \frac{\partial^2 z}{\partial x \partial y} \to f_{xy} \qquad \frac{\partial^2 z}{\partial y^2} \to f_{yy}$$

Ex – find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$z = 2x^2y + xy^2 + 5xy$$

soln –

$$\frac{\partial z}{\partial x} = 2y \cdot 2x + y^2 \cdot 1 + 5y \cdot 1$$

(Treat y as constant)

$$= 4xy + y^2 + 5y$$

$$\frac{\partial z}{\partial y} = 2x^2 \cdot 1 + x \cdot 2y + 5x \cdot 1$$

(Treat x as constant)

$$= 2x^2 + 2xy + 5x$$

Ex – $z = 2xy + x^2$

soln –

$$\frac{\partial z}{\partial x} = 2y \cdot 1 + 2x$$

$$= 2y + 2x$$

$$\frac{\partial z}{\partial y} = 2x + 0$$

$$= 2x$$

Successive differentiation

$$y = x^5$$

$\frac{dy}{dx} = 5x^4$ — 1st Order derivative

$\frac{d^2y}{dx^2} = 20x^3$ — 2nd Order derivative

$\frac{d^3y}{dx^3} = 60x^2$ — 3rd Order derivative

$\frac{d^4y}{dx^4} = 120x$ — 4th Order derivative

$\frac{d^5y}{dx^5} = 120$ — 5th Order derivative

$\frac{d^6y}{dx^6} = 0$

Defination

Let 'f' be a differentiable function of x, then the derivative of $f(x)$ may determine another differentiable function of x. The new function $f'(x)$ is called the 1st derivative of f.

If $f'(x)$ is differentiable, we can find it's derived function $f''(x)$ and call it the derived function of 2nd order. The process of finding higher order derivatives is called successive differentiation.

Q. If $z = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$ Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

soln - $z = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2} \cdot \frac{\partial}{\partial x}(x^2+y^2) + \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{x^2+y^2} 2x + \frac{x^2}{x^2+y^2} y \left(-\frac{1}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} + \frac{x^2 y}{x^2+y^2}\left(\frac{-1}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2}$$

$$= \frac{2x-y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2x-y}{x^2+y^2}\right)$$

$$= \frac{x^2+y^2 \frac{\partial}{\partial x}(2x-y) - (2x-y)\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)2 - (2x-y)2x}{(x^2+y^2)^2}$$

$$= \frac{2(x^2+y^2) - 2x(2x-y)}{(x^2+y^2)^2} = \left[\frac{2x^2+2y^2-4x^2+2xy}{(x^2+y^2)^2}\right]$$

$$= 2\left(\frac{x^2+y^2-2x^2+xy}{(x^2+y^2)^2}\right)$$

Now, $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)\right)$

$$= \frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial y}(x^2+y^2) + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{x^2+y^2}\cdot 2y + \frac{x^2}{x^2+y^2}\cdot\frac{1}{x}$$

$$= \frac{2y}{x^2+y^2} + \frac{1x}{x^2+y^2}$$

$$= \frac{2y+x}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{2y+x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial y}(2y+x) - (2y+x)\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)2 - 2y+x\,(2y)}{(x^2+y^2)^2}$$

$$= \frac{2(x^2+y^2) - 2y(2y+x)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4y^2-2xy}{(x^2+y^2)^2}$$

$$= \frac{2(x^2-y^2-xy)}{(x^2+y^2)^2}$$

Now, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$

$$= \frac{2(x^2+y^2-2x^2+xy)}{(x^2+y^2)^2} + \frac{2(x^2-y^2-xy)}{(x^2+y^2)^2}$$

$$= \frac{2(-x^2+y^2+xy)}{(x^2+y^2)^2} + 2\left(\frac{x^2-y^2-xy}{(x^2+y^2)^2}\right)$$

$= 0$ Proved

2nd order partial differentiation

$$z = f(x, y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = z_{xx} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = z_{yx} = f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = z_{xy} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = z_{yy} = f_{yy}$$

If we differentiate $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ with respect to x or y then we get higher order partial derivatives.

Ex- find f_{xx} and f_{xy}, f_{yx} where

$$f(x, y) = x^2 + y^3 + 3xy$$

Soln- $f_x = \frac{\partial}{\partial x}(x^3 + y^3 + 3xy)$

$$= 3x^2 + 0 + 3y \cdot 1$$

$$= 3x^2 + 3y$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x)$$

$$= \frac{\partial}{\partial x}(3x^2 + 3y)$$

$$= 6x + 0 = 6x$$

$$f_y = \frac{\partial}{\partial y}(x^3 + y^3 + 3xy)$$

$$= 0 + 3y^2 + 3x \cdot 1$$

$$= 3y^2 + 3x$$

$$f_{xy} = \frac{\partial}{\partial x}(f_y)$$

$$= \frac{\partial}{\partial x}(3y^2 + 3x)$$

$$= 0 + 3 = 3$$

$$f_{yx} = \frac{\partial}{\partial x}(f_x)$$

$$= \frac{\partial}{\partial x}(3x^2 + 3y)$$

$$= 0 + 3 = 3$$

Q. If $z = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$ Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

soln -

$$z = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2} \cdot \frac{\partial}{\partial x}(x^2+y^2) + \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{x^2+y^2}\, 2x + \frac{x^2}{x^2+y^2}\, y\left(-\frac{1}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} + \frac{\cancel{x^2}\, y}{x^2+y^2}\left(-\frac{1}{\cancel{x^2}}\right)$$

$$= \frac{2x-y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2x-y}{x^2+y^2}\right)$$

$$= \frac{x^2+y^2 \frac{\partial}{\partial x}(2x-y) - (2x-y)\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)2-(2x-y)2x}{(x^2+y^2)^2}$$

$$= \frac{2(x^2+y^2-2x^2+xy)}{(x^2+y^2)^2} = \frac{2(-x^2+y^2+xy)}{(x^2+y^2)^2}$$

Now, $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\log(x^2+y^2)+\tan^{-1}\left(\frac{y}{x}\right)\right)$

$$= \frac{1}{x^2+y^2}\frac{\partial}{\partial y}(x^2+y^2) + \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{x^2}{x^2+y^2}\cdot\frac{1}{x}$$

$$= \frac{2y+x}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{2y+x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)2-(2y+x)2y}{(x^2+y^2)^2}$$

$$= \frac{2(x^2+y^2-2y^2-xy)}{(x^2+y^2)^2}$$

$$= \frac{2(x^2-y^2-xy)}{(x^2+y^2)^2}$$

Now, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$

$$= \frac{2(-x^2+y^2+xy)}{(x^2+y^2)^2} + \frac{2(x^2-y^2-xy)}{(x^2+y^2)^2}$$

$$= \frac{2(\cancel{-x^2}+\cancel{y^2}+\cancel{xy}+\cancel{x^2}-\cancel{y^2}-\cancel{xy})}{(x^2+y^2)^2}$$

$= 0 =$ RHS $\boxed{\text{Proved}}$

INTEGRATION (Antidifferentiation)

If $g(x)$ is the derivative of $f(x)$, then $f(x)$ is called the antiderivative or integral of $g(x)$.

i.e $\int g(x)\,dx = f(x) + c$

→ Integration constant

→ with respect to x we integrate or Variable of integration

→ Integrand

Ex – $\frac{d}{dx}(\sin x) = \cos x$

$\int \cos x\,dx = \sin x + c$

$\frac{d}{dx}(\cos x) = -\sin x$

$\int \sin x\,dx = -\cos x + c$

* $\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\sin x + 2) = \cos x$

$\frac{d}{dx}(\sin x + 7) = \cos x$

$\frac{d}{dx}(\sin x + c) = \cos x$

Types of integration

i) Indefinite integration $\left(\int g(x)\,dx\right)$

ii) Definite integration $\left(\int_a^b g(x)\,dx\right)$

Integration formulae

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

2) $\int \frac{1}{x} dx = \ln|x| + c$

3) $\int \cos x \, dx = \sin x + c$

4) $\int \sin x \, dx = -\cos x + c$

5) $\int \sec^2 x \, dx = \tan x + c$

6) $\int \sec x \tan x \, dx = \sec x + c$

7) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

8) $\int e^x dx = e^x + c$

9) $\int a^x dx = \frac{a^x}{\ln a} + c$

10) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$ or $-\cos^{-1} x + c$

11) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$ or $-\cot^{-1} x + c$

12) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$ or $-\operatorname{cosec}^{-1} x + c$

Algebra of integrals

1) $\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$

2) $\int \lambda f(x) dx = \lambda \int f(x) dx$, λ is a constant.

Integrate

$\int (x^6 + x^2 + x + 1) dx$

$= \int x^6 dx + \int x^2 dx + \int x dx + \int 1 dx$

$$= \frac{x^{6+1}}{6+1} + \frac{x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} + x + c$$

$$= \frac{x^7}{7} + \frac{x^3}{3} + \frac{x^2}{2} + x + c \quad \text{Ans}$$

Ex - Integrete

$$\int \left(4\cos x - 3e^x + \frac{2}{\sqrt{1-x^2}}\right) dx$$

Soln -

$$\int 4\cos x\, dx - \int 3e^x dx + \int \frac{2}{\sqrt{1-x^2}} dx$$

$$= 4\int \cos x\, dx - 3\int e^x dx + 2\int \frac{1}{\sqrt{1-x^2}} dx$$

$$= 4\sin x - 3e^x + 2\sin^{-1}x + c \quad \text{Ans}$$

ENGINEERING MATHEMATICS

Ex - $\int e^{3x} dx$

soln - $\cancel{\int (e^{x})^{3} dx}$

$= \int (e^3)^x dx$

$= \dfrac{(e^3)^x}{\ln e^3} + c$

$= \dfrac{e^{3x}}{3} + c$

Ex - $\int \frac{\sin^2 x}{1+\cos x} dx$ || $\sin^2 x + \cos^2 x = 1$
$\sin^2 x = 1 - \cos^2 x$

Soln - $\int \frac{1-\cos^2 x}{1+\cos x} dx \longrightarrow$ use $a^2 - b^2 = (a+b)(a-b)$

$= \int \frac{\cancel{(1+\cos x)}(1-\cos x)}{\cancel{(1+\cos x)}} dx$

$= \int (1-\cos x)\, dx$

$= \int 1\, dx - \int \cos x\, dx$

$= x - \sin x + c$

Q. $\int \sqrt{1+\sin 2x}\, dx$ || $\sin 2x = 2\sin x \cdot \cos x$

soln - $\int \sqrt{\sin^2 x + \cos^2 x + 2 \cdot \sin x \cdot \cos x}\, dx$

$= \int \sqrt{(\sin x + \cos x)^2}\, dx$

$= \int (\sin x + \cos x)\, dx$

$= -\cos x + \sin x + c$ Ans

Q. $\int \frac{\cos 2x}{\cos x + \sin x} dx$ || $\cos 2x = \cos^2 x - \sin^2 x$

Soln -

$\int \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x} dx$

$$= \int \frac{\cancel{(\cos x + \sin x)}(\cos x - \sin x)}{\cancel{\cos x + \sin x}} dx$$

$$= \int (\cos x - \sin x)\, dx$$

$$= \sin x + \cos x + C \quad \text{Ans}$$

Q1 $\int \sqrt{1 - \cos 2x}\, dx$

Q2. $\int \sqrt{1 + \cos 2x}\, dx$

Soln 1 - $\int \sqrt{1 - \cos 2x}\, dx$

$$= \int \sqrt{\sin^2 x + \cos^2 x - (\cos^2 x - \sin^2 x)}\, dx$$

$$= \int \sqrt{\sin^2 x + \cancel{\cos^2 x} - \cancel{\cos^2 x} + \sin^2 x}\, dx$$

$$= \int \sqrt{2 \sin^2 x}\, dx$$

$$= \sqrt{2} \int \sin x\, dx$$

$$= -\sqrt{2} \cos x + C \quad \text{Ans}$$

Soln 2 - $\int \sqrt{1 + \cos 2x}\, dx$

$$= \int \sqrt{\sin^2 x + \cos^2 x + (\cos^2 x - \sin^2 x)}\, dx$$

$$= \int \sqrt{\cancel{\sin^2 x} + \cos^2 x + \cos^2 x - \cancel{\sin^2 x}}\, dx$$

$$= \int \sqrt{2 \cos^2 x}\, dx = \sqrt{2} \int \cos x\, dx = \sqrt{2} \sin x + C$$

Integration by substitution

When the integrand isn't in a standard form it can sometimes be transformed to integrable form by a suitable substitution.

* $\int f(g(x))\, g'(x)\, dx$ can be converted to

$$= \int f(u)\, du$$

$$= F(u) + k$$

$$= F(g(x)) + k$$

Let

$u = g(x)$

$\frac{du}{dx} = g'(x)$

$du = g'(x)\, dx$

Ex - $\int (ax+b)^n\, dx$

Soln -

Let,

$u = ax+b$

$du = a\,dx$

$\frac{du}{a} = dx$

$$= \int u^n \frac{du}{a}$$

$$= \frac{1}{a} \int u^n\, du$$

$$= \frac{1}{a} \frac{u^{n+1}}{n+1} + c$$

$$= \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + c$$

Ex - $\int \sin(ax+b)\, dx$

Soln - $\int \sin u \cdot \frac{du}{a}$

Let,

$u = ax+b$

$du = a\,dx$

$\frac{du}{a} = dx$

$$= \frac{1}{a} \int \sin u \cdot du$$

$$= \frac{1}{a} (-\cos u) + c$$

$$= -\frac{1}{a} \cos(ax+b) + c$$

* $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

* $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+c$

* $\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a}+c$

$$\int e^{5x}\,dx = \frac{e^{5x}}{5}+c$$

$$\int e^{7x+3}\,dx = \frac{e^{7x+3}}{7}+c$$

Q. $\int 2e^{\tan^2 x}\tan x \cdot \sec^2 x\,dx$

soln – $\int e^u\,du$

$u = \tan^2 x$

$du = 2\tan x \cdot \sec^2 x\,dx$

$= e^u + c$

$= e^{\tan^2 x} + c$

ENGINEERING MATHEMATICS

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* $$\int \frac{g'(x)}{g(x)}\,dx$$

Let $u = g(x)$

$$\frac{du}{dx} = g'(x)$$

$\Rightarrow du = g'(x)\,dx$

$= \int \frac{1}{u}\,du$

$= \ln|u| + c$

$= \ln|g(x)| + c$

Imp. Q. Integrate $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$

Soln – Let $u = e^x - e^{-x}$

$du = \{e^x - (-e^{-x})\}\,dx$

$= (e^x + e^{-x})\,dx$

$= \int \frac{1}{u}\,du$

$= \ln|u| + c$

$= \ln|e^x - e^{-x}| + c$ Ans

* $\int \frac{7dx}{2-3x}$

Soln – Let $u = 2 - 3x$

$du = -3dx$

$-\frac{du}{3} = dx$

So, $\int \frac{7dx}{2-3x} = 7\int \frac{dx}{2-3x}$

$= 7\int \frac{1}{u}\left(-\frac{du}{3}\right)$

$= -\frac{7}{3}\int \frac{1}{u}\,du$

$= -\frac{7}{3}\ln|u| + c$

$= -\frac{7}{3}\ln|2-3x| + c$ Ans

Q. $\int \sin^7 x \cos x \, dx$

Soln - Let $u = \sin x$

$du = \cos x \, dx$

$= \int u^7 dx$

$= \frac{u^8}{8} + c$

$= \frac{\sin^8 x}{8} + c$ Ans

* $\int 2a^{x^2} x \, dx$

Soln - Let

$u = x^2$

$du = 2x \, dx$

So,

$= \int a^u du$

$= \frac{a^u}{\ln a} + c$

$= \frac{a^{x^2}}{\ln a} + c$ Ans

Imp

* $\int \cot x \, dx = \ln|\sin x| + c$

Proof - $\int \frac{\cos x}{\sin x} dx$ Let $u = \sin x$

$du = \cos x \, dx$

$= \int \frac{1}{u} du$

$= \ln|u| + c = \ln|\sin x| + c$

* $\int \tan x \, dx = \ln|\sec x| + c$

Proof - $\int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$

Let $u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= \int \frac{1}{u}(-du)$

$= -\int \frac{1}{u} du$

$= -\ln|u| + c$

$= -\ln|\cos x| + c$

$= \ln|\cos x^{-1}| + c$ // $\cos^{-1} = \frac{1}{\cos x} = \sec x$

$= \ln|\sec x| + c$ Ans

* $\int \sec x \, dx = \ln|\sec x + \tan x| + c$

Proof - $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

Let, $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) dx$

$= \int \frac{1}{u} du$

$= \ln|u| + c$

$= \ln|\sec x + \tan x| + c$

Homework

$\int \operatorname{cosec} x \, dx = \ln|\operatorname{cosec} x - \cot x| + c$

Q. $\int e^x \tan e^x \, dx$

soln - Let $u = e^x$

$du = e^x dx$

$= \int \tan u \, du$

$= \ln|\sec u| + c$

$= \ln|\sec e^x| + c$

Q. $\int \frac{\operatorname{cosec}^2 x}{1 + \cot x} dx$

soln - Let $u = 1 + \cot x$

$du = -\operatorname{cosec}^2 x \, dx$

$-du = \operatorname{cosec}^2 x$

$= -\int \frac{1}{u} du$

$= -\ln|u| + c$

$= -\ln|1 + \cot x| + c$

Integration by parts

To integrate the function of two functions, we use the rule 'integration by parts'.

$$\int \underset{\text{1st function}}{u}\overset{\text{2nd function}}{v} dx = v\int u\,dx - \int \left(\int u\,dx\right) \frac{d}{dx}(v)\,dx$$

= 2nd function × Integral of 1st function − Integral of (Integral of 1st × derivative of 2nd)

Use 'ETALI' to choose 1st function

E : Exponential function

T : Trigonometric function

A : Algebraic function

L : Logarithm function

I : Inverse function

Integrand	1st function	2nd function
$x^n e^x$	e^x	x^n
$x^n \cos x$	$\cos x$	x^n
$x^n \ln x$	x^n	$\ln x$
$x^n \tan^{-1} x$	x^n	$\tan^{-1} x$
$\tan^{-1} x$	1	$\tan^{-1} x$

Ex - $\int \underset{v}{x} \underset{u}{\cos x}\, dx$

Soln - 1st function → $\cos x$

2nd function → x

$$= x \int \cos x\, dx - \int \left(\int \cos x\, dx \right) \frac{dx}{dx}\, dx$$

$$= x \sin x - \int \sin x \; 1\, dx$$

$$= x \sin x - (-\cos x) + c$$

$$= x \sin x + \cos x + c$$ Ans

Ex - $\int \underset{v}{x^2} \underset{u}{e^x}\, dx$

Soln - 1st function → e^x

2nd function → x^2

$$= x^2 \int e^x dx - \int \left(\int e^x dx \right) \frac{d}{dx}(x^2)\, dx$$

$$= x^2 e^x - \int e^x \, 2x \, dx$$

$$= x^2 e^x - 2 \int e^x x \, dx$$

$$= x^2 e^x - 2 \left[x \int e^x dx - \int \left(\int e^x dx \right) \frac{dx}{dx} \, dx \right]$$

$$= x^2 e^x - 2 \left\{ x e^x - \int e^x \, 1 \, dx \right\}$$

$$= x^2 e^x - 2x e^x + 2e^x + c$$

$$= e^x (x^2 - 2x + 2) + c$$ Ans

* when $\sin^{-1}x$, $\cot^{-1}x$, $\tan^{-1}x$ etc or $\log x$ is precent alone in the integrand take 1 as the 1^{st} function.

Ex - $\int \tan^{-1}x \, dx$

Soln -

$$\int \underbrace{1}_{1st} \underbrace{\tan^{-1}x}_{2nd} \, dx$$

$$= \tan^{-1}x \int 1\,dx - \int \left(\int 1\,dx\right) \frac{d}{dx}(\tan^{-1}x)\,dx$$

$$= \tan^{-1}x \cdot x - \int x \cdot \frac{1}{1+x^2}\,dx$$

$$= x\tan^{-1} - \int \frac{x}{1+x^2}\,dx$$

* Let, $u = 1+x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= x\tan^{-1}x - \int \frac{1}{u} \cdot \frac{du}{2}$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{1}{u}\,du$$

$$= x\tan^{-1}x - \frac{1}{2}\ln|u| + c$$

$$= x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + c$$ Ans

* $\int \log x\,dx$

Soln - $\int \underbrace{1}_{1st} \underbrace{\log x}_{2nd}\,dx$

$$= \log x \int 1\,dx - \int \int 1\,dx \frac{d}{dx}(\log x)\,dx$$

$$= \cancel{\log x}\; x\log x - \int \cancel{x} \cdot \frac{1}{\cancel{x}}\,dx$$

$$= x\log x - \int 1\,dx$$

$$= x\log x - x + c = x(\log x - 1) + c$$ Ans

Homework

$\int (\ln x)^2 dx$

* $\int e^{ax} \cos bx \, dx$

soln - $\cos bx \int e^{ax} dx - \int \left(\int e^{ax} dx \right) \frac{d}{x} \cos bx \, dx$

$= \cos bx \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} (-b \sin bx) dx$

$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \, dx$

$= \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \left[\sin bx \cdot \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} b \cos bx \, dx \right]$

$= \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx \, dx$

$\Rightarrow \int e^{ax} \cos bx \, dx + \frac{b^2}{a^2} \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a} \cos bx + \frac{b}{a^2} e^{ax} \sin bx$

$\Rightarrow \left(\int e^{ax} \cos bx \, dx \right) \left(1 + \frac{b^2}{a^2} \right) = e^{ax} \left[\frac{\cos bx}{a} + \frac{b}{a^2} \sin bx + C \right.$

$\Rightarrow \int e^{ax} \cos bx \, dx = \frac{a^2}{a^2+b^2} \cdot e^{ax} \left\{ \frac{\cos bx}{a} + \frac{b}{a^2} \sin bx \right\} + \frac{Ca^2}{a^2+b^2}$

$= \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) + K$ Ans.

Ex - $\int e^{3x} \cos 2x \, dx$

Ans - $\frac{e^{3x}}{3^2+2^2} (3 \cos 2x + 2 \sin 2x) + K$

$= \frac{e^{3x}}{13} (3 \cos 2x + 2 \sin 2x) + K$ Ans

* $\int e^{ax} \sin bx \, dx$

$$= \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx) + k$$

Ex - $\int e^{2x} \sin x \, dx$

$$= \frac{e^{2x}}{2^2+1^2}(2\sin x - \cos x) + k$$

$$= \frac{e^{2x}}{5}(2\sin x - \cos x) + k \quad \text{Ans}$$

* $\int \sqrt{a^2-x^2}\, dx$ Take 1 as 1st function

soln - $\int 1 \cdot \sqrt{a^2-x^2}\, dx$

$$= \sqrt{a^2-x^2}\int 1\,dx - \int\left(\int 1\,dx\right)\frac{d}{dx}\left(\sqrt{a^2-x^2}\right)dx$$

$$= x\sqrt{a^2-x^2} - \int x \cdot \frac{1}{\cancel{2}\sqrt{a^2-x^2}}(-\cancel{2}x)\,dx$$

$$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} + \int \frac{a^2}{\sqrt{a^2-x^2}}\,dx - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} + a^2\int \frac{1}{\sqrt{a^2-x^2}}\,dx - \underline{\int\sqrt{a^2-x^2}\,dx}$$ Take to LHS

$$= 2\int\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a} + c$$

$$= \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + k \quad \text{Ans}$$

$k = \frac{c}{2}$

Ex- $\int \sqrt{9-x^2}\, dx$

Soln - $\int \sqrt{3^2 - x^2}\, dx$

$= \frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + K$ Ans

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Definite integration

Integration is a process of summation. In this case the Integration is called definite integration.

$$h = \frac{b-a}{n}$$

$$\int_a^b f(x)\,dx \rightarrow \text{Definite integration}$$

$b \rightarrow$ Upper limit

$a \rightarrow$ Lower limit

Fundamental theorem of integral calculus:

$$\int_a^b f(x)\,dx = \left[f(x)\right]_a^b$$

$$= F(b) - F(a)$$

Ex - $\int_1^2 x^2\,dx$

$$= \left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3}\left[2^3 - 1^3\right]$$

$$= \frac{1}{3}(8-1) = \frac{7}{3} \quad \text{Ans}$$

Ex - $\int_0^1 \frac{dx}{1+x^2}$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4} \quad \text{Ans}$$

Elementary properties of definite integrals

(i) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(ii) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(z)\,dz$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a < c < b$

Ex - $\int_1^4 x^2\,dx = \left.\frac{x^3}{3}\right|_1^4 = \frac{64-1}{3} = \frac{63}{3} = 21$

OR

$= \int_1^2 x^2\,dx + \int_2^3 x^2\,dx + \int_3^4 x^2\,dx$

$= \left.\frac{x^3}{3}\right|_1^2 + \left.\frac{x^3}{3}\right|_2^3 + \left.\frac{x^3}{3}\right|_3^4$

$= \frac{7}{3} + \frac{19}{3} + \frac{37}{3} = \frac{63}{3} = 21$

AREA UNDER PLANE CURVES

Area enclosed by a curve and x - axis

$$A = \int_{x=a}^{b} f(x)\,dx$$

Ex - Find the area of the region enclosed by $y = 9 - x^2$, $y = 0$ and the ordinates $x = 0$ and $x = 2$.

Soln -

$$A = \int_0^2 f(x)\,dx$$

$$= \int_0^2 (9 - x^2)\,dx$$

$$= \int_0^2 9\,dx - \int_0^2 x^2\,dx$$

$$= 9\,[x]_0^2 - \left[\frac{x^3}{3}\right]_0^2$$

$$= 9(2-0) - \frac{1}{3}(8-0)$$

$$= 18 - \frac{1}{3} \times 8 = \frac{46}{3} \text{ sq. unit}$$

Area of circle with the centre at origin

Equation of circle with centre at origin.

$$x^2 + y^2 = r^2$$

centre . $(0,0)$

radius $= r$

Q. Find the area of circle $x^2 + y^2 = a^2$

Ans - $x^2 + y^2 = a^2$

Centre $= (0, 0)$

radius $= a$

$$x^2 + y^2 = a^2$$

$$y^2 = a^2 - x^2$$

$$y = \sqrt{a^2 - x^2}$$

So, Area in 1st quadrant

$$= \int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{2} \right]_0^a$$

$$= \frac{a^2}{2} \sin^{-1}(1)$$

$$= \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{a^2 \pi}{4} = \frac{\pi a^2}{4}$$

Area of circle

$$= 4 \times \frac{\pi a^2}{4}$$

$$= \pi a^2 \text{ sq. unit}$$

DIFFERENTIAL EQUATIONS

Equations involving derivative or differentials of dependent variable with respect to independent variable are known as differential equations.

Dependent variable

$$\frac{dy}{dx} + 2xy = x^2$$

Independent variable

$\rightarrow$ $x\,dy + y\,dx = 0$

Differential equation

ODE (Ordinary differential equation)

PDE (partial differential equation)

Order of a differential equation

The order of the highest order derivative Occuring in the equation is known as order of the differential equation.

Degree of a differential equation

This is the highest positive integral power of the derivative that determines the order of the equation.

It is determined after the equation is cleared of fractional indias with regard to all derivatives involved and after the denominators are cated of derivatives.

Ex - $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + xy = 0$$

soln -

Order = 2

Degree = 1

Ex - $\left(\frac{dy}{dx}\right)^4 + y^5 = \frac{d^3y}{dx^3}$

soln - Order = 3

degree = 1

Ex - $a\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$

soln - Squaring both sides,

$$a^2\left(\frac{d^2y}{dx^2}\right)^2 = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3$$

Order - 2

Degree - 2

Q. $\ln\left(\frac{d^2y}{dx^2}\right) = y$

soln - $\frac{d^2y}{dx^2} = e^y$

Order - 2

Degree - 1

Q. $$\frac{\frac{dy}{dt}}{y + \frac{dy}{dt}} = \frac{yt}{\frac{dy}{dt}}$$

$\Rightarrow \left(\frac{dy}{dt}\right)^2 = y + \left(y + \frac{dy}{dt}\right)$

Order - 1

Degree - 2

Q. $\dfrac{d^2y}{du^2} = \dfrac{3y + \dfrac{dy}{du}}{\sqrt{\dfrac{d^2y}{du^2}}}$

soln - $\left(\dfrac{d^2y}{du^2}\right)^{\frac{3}{2}} = 3y + \dfrac{dy}{du}$

Squaring both sides,

$$\left(\dfrac{d^2y}{du^2}\right)^3 = \left(3y + \dfrac{dy}{du}\right)^2$$

Order - 2

Degree - 3

Solution of differential equation by separation of variable method

The process of collecting or functions of (x) with dx and all functions of y with dy is known as the process of separation of variable.

Q. $\dfrac{dy}{dx} = f(x)$

$\Rightarrow \int dy = \int f(x)\,dx$

$\Rightarrow y = F(x) + c \rightarrow$ General solution

Q. $\dfrac{dy}{dx} = f(y)$

$\Rightarrow \int \dfrac{dy}{f(y)} = \int dx$

$\Rightarrow \int \dfrac{dy}{f(y)} = x + c$

Ex - $\frac{dy}{dx} = x^2 + 2x + 5$

soln - $dy = (x^2+2x+5)\,dx$

Now integrating both sides,

$$\int dy = \int (x^2+2x+5)\,dx$$

$$y = \frac{x^3}{3} + 2\cdot\frac{x^2}{5} + 5\cdot x + c$$

$$y = \frac{x^3}{3} + x^2 + 5x + c$$

Ex - $\frac{dy}{dx} = \tan y$

soln - $\frac{dy}{\tan y} = dx$

$\Rightarrow \cot y\,dy = dx$

Now integrating both sides,

$$\int \cot y\,dy = \int dx$$

$\Rightarrow \ln|\sin y| = x + c$

$\Rightarrow \sin y = e^{x+c}$

$= e^x \cdot e^c$

$= Ae^x \quad // \; A = e^c$ Ans

Ex - $\frac{dy}{dx} = \frac{2y}{x^2+1}$

soln - $\frac{dy}{y} = \frac{2}{x^2+1}\,dx$

Now integrating both sides

$$\int \frac{dy}{y} = 2\int \frac{1}{x^2+1}\,dx$$

$\Rightarrow \ln y = 2\tan^{-1} x + c$ Ans

Ex - $\frac{d^2y}{dx^2} = 12x^2 + 2x$

soln - $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Let $\frac{dy}{dx} = P$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{dP}{dx}$

$\Rightarrow \frac{dP}{dx^2} = 12x^2 + 2x$

$\Rightarrow \int dP = \int (12x^2 + 2x)dx$

$\Rightarrow P = \cancel{12}^{4} \cdot \frac{x^3}{\cancel{3}} + \cancel{2} \cdot \frac{x^2}{\cancel{2}} + C$

$\Rightarrow P = 4x^3 + x^2 + C$

$\Rightarrow \frac{dy}{dx} = 4x^3 + x^2 + C$

$\Rightarrow \int dy = \int (4x^3 + x^2 + C)\, dx$

$\Rightarrow y = 4 \cdot \frac{x^4}{4} + \frac{x^3}{3} + Cx + D$

$\Rightarrow y = x^4 + \frac{x^3}{3} + Cx + D$ Ans

Ex - $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Let $\frac{dy}{dx} = P$

$\Rightarrow \frac{d^2y}{dx} = \frac{dP}{dx}$

$\frac{dP}{dx} = \sin x - \cos x$

$\Rightarrow \int dp = \int (\sin x - \cos x)\, dx$

$\Rightarrow p = -\cos x - \sin x + c$

$$\frac{dy}{dx} = -\cos x - \sin x + c$$

$$\Rightarrow \int dy = \int (-\cos x - \sin x + c)\, dx$$

$$\Rightarrow y = -\sin x + \cos x + cx + D \quad \text{Ans}$$

Ex - $\frac{dy}{dx} = \cos x$, $y(0) = 2$

i.e when $x = 0, y = 0$

$$\int dy = \int \cos x\, dx$$

$$\Rightarrow y = \sin x + c \quad \text{——(1)}$$

By given conditions $y(0) = 2$

$\Rightarrow 2 = \sin 0 + c$

$\Rightarrow 2 = 0 + c$

$\Rightarrow c = 2$

So, (1) becomes

$y = \sin x + 2$ (Particular solution)

Linear differential equation

A differential equation is said to be linear if the dependent variable and it's differential co-efficients occuring in the equation are of degree one and are not multiplied together.

Ex - $\frac{dy}{dx} + xy = 0$ linear equation

$\frac{d^2y}{dx^2} + xy^2 = x^3$ Non-linear

$\frac{d^2y}{dx^2} + y \cdot \frac{dy}{dx} = x^2$ Non-linear

$\frac{dy}{dx} + P(x)y = Q(x)$ → linear differential equation

Where $P(x)$, $Q(x)$ are function of x.

How to solve

Integrating factor

$$I.F = e^{\int P dx}$$

Solution is

$$y\, I.F = \int I.F \cdot Q(x)\, dx + c$$

Ex – $(1+x^2)\frac{dy}{dx} + 2xy - x^3 = 0$

Soln – $\frac{dy}{dx} + \frac{2xy}{1+x^2} - \frac{x^3}{1+x^2} = 0$

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2} \cdot y = \frac{x^3}{1+x^2}$$

Which is a linear equation

Here, $P = \frac{2x}{1+x^2}$, $Q = \frac{x^3}{1+x^2}$

$$I.F = e^{\int P dx}$$

$$= e^{\int \frac{2x}{1+x^2} dx}$$

$$= e^{\int \frac{du}{u}}$$

$u = 1+x^2$

$du = 2x\,dx$

$$= e^{\ln u}$$

$$= e^{\ln(1+x^2)}$$

$$= 1+x^2$$

$$\Rightarrow y \cdot (1+x^2) = \int (1+x^2) \frac{x^3}{(1+x^2)} dx + c$$

$$= \int x^3 dx + c$$

$$= \frac{x^4}{4} + c$$

$$\Rightarrow y = \frac{x^4}{4(1+x^2)} + \frac{c}{1+x^2}$$ Ans