LECTURES NOTE

SUB: ENGINEERING MATHEMATICS III

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COMPLEX NUMBER

$\Rightarrow$ Let's consider a quadratic equation

$$ax^2+bx+c=0\ ,\quad a,b,c \in R$$

e.g. $x^2+1=0\ ,\ x \in R$

$\Rightarrow x^2=-1$

$\Rightarrow x=\sqrt{-1}$ (so there is no. solution)

so, let $\sqrt{-1} = i$ (iota) imaginary no. [does'nt exist]

and value of $\boxed{i^2=-1}$

Example of imaginary no.s $\rightarrow 3i, -4i, \sqrt{3}i, \frac{i}{2}$, etc.

i.e. imaginary numbers are of the form iR.

$\Rightarrow$ Now if we form numbers like $\underline{R+iR}$ $\downarrow$ it is called complex Number.

having both real part and imaginary part.

Basically complex Numbers are defined as

$$\boxed{C = \{ z \mid z = x+iy,\ \text{where } x,y \in R \}}$$

In complex Numbers $Z = x+iy\ ,\ x,y \in R$

where x = Real part of Z

y = Imaginary part of Z

For example

(i) $z = 3 - 4i$

(ii) $z = -2 + 5i$

(iii) $z = \sqrt{2} + i$

(iv) $z = 1 + 3i$

Now $z = 3i$ it is also a complex number where real part is zero.

** And here $z = 3i$ is said to be purely imaginary as $Re(z) = 0$

Again $z = -7$ is also a complex Number. where imaginary part is zero.

** So $z = -7$ is said to be purely real numbers.

Now

$\Rightarrow$ We have $i = \sqrt{-1}$

$$\boxed{i^2 = -1} \text{ ——— } (*)$$

$$i^3 = i^2 \cdot i$$
$$= -1 \cdot i$$
$$\Rightarrow \boxed{i^3 = -i} \text{ ——— } (*)$$

$$i^4 = i^2 \cdot i^2$$
$$= (-1) \cdot (-1)$$
$$\Rightarrow \boxed{i^4 = 1} \text{ ——— } (*)$$

$i^5 = i^4 \cdot i = i$

$i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = -1$

$i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

$i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1$

$i^9 = i^{4\times 2} \cdot i$
$= (i^4)^2 \cdot i$
$= (1)^2 \cdot i = i$

$i^{10} = i^{4\times 2} \cdot i^2$
$= (i^4)^2 \cdot (-1) = 1 \cdot (-1)$
$= -1$

So we got

*
$$\left.\begin{array}{l} i = i^5 = i^9 = i^{13} = \dots\dots = i^{4n+1} \\ i^2 = i^6 = i^{10} = i^{14} = \dots\dots = i^{4n+2} \\ i^3 = i^7 = i^{11} = i^{15} = \dots\dots = i^{4n+3} \\ i^4 = i^8 = i^{12} = i^{16} = \dots\dots = i^{4n} \end{array}\right\}, n \in \mathbb{N}$$

* If there is a question find i^{50}

Method 1 $\quad i^{50} = i^{4\times 12+2} = i^2 = -1$

Method 2 are $\quad i^{50} = i^{4\times 12+2}$

$= i^{4\times 12} \cdot i^2$

$= (i^4)^{12} (-1)$

$= (1)^{12} (-1)$

$= 1(-1)$

$= -1$

imaginary Numbers

OPERATIONS ON COMPLEX NUMBER $(+, -, \times, \div)$

Let $z_1 = a + ib$

$z_2 = c + id$ be two complex Numbers.

Then

(i) $z_1 + z_2 = (a+ib) + (c+id)$

$= (a+c) + (ib+id)$

$= (a+c) + i(b+d)$

(ii) $z_1 - z_2 = (a+ib) - (c+id)$

$= (a-c) + i(b-d)$

(iii) $z_1 z_2 = (a+ib)(c+id)$

$= ac + iad + ibc + i^2 bd$

$= ac + iad + ibc - bd$

$= (ac-bd) + i(ad+bc)$

CONJUGATE AND MODULUS OF A COMPLEX NUMBER

Let $z = x + iy$ be a complex Number.

Then **Modulus** of z is denoted by $|z|$

and is defined as $|z| = |x+iy| = \sqrt{x^2+y^2}$

So $\boxed{|z| = \sqrt{x^2+y^2}}$

And **Conjugate** of z is denoted by $\overline{z}$

and is defined as $\boxed{\overline{z} = x - iy}$

* That means conjugate is obtained by changing the sign of imaginary part.

for example

Q.1 find conjugate and Modulus of

$z = 3 + 4i$

Now given $z = 3+4i$, here $x = 3, y = 3$

$\overline{z} = 3 - 4i$

$|z| = \sqrt{(3)^2 + (4)^2}$

$= \sqrt{9+16} = \sqrt{25} = 5$

Q.4 Express in a+ib form i.e $\frac{2+3i}{4-2i}$

Ans
$$Z = \frac{2+3i}{4-2i} = \frac{(2+3i)(4+2i)}{(4-2i)(4+2i)}$$

$$= \frac{8+4i+12i+6i^2}{(4)^2-(2i)^2}$$

$$= \frac{8+16i-6}{16-4i^2} \quad (\because i^2=-1)$$

$$= \frac{2+16i}{16-4(-1)}$$

$$= \frac{2+16i}{16+4} = \frac{2+16i}{20}$$

$$= \frac{2}{20} + \frac{16}{20}i$$

$$= \frac{1}{10} + 4/5\, i$$

Q.5 find the conjugate of the complex no. $\frac{1}{3+4i}$

Sol^n $Z = \frac{1}{3+4i}$

$$= \frac{1(3-4i)}{(3+4i)(3-4i)} = \frac{3-4i}{(3)^2-(4i)^2} = \frac{3-4i}{9-16(-1)}$$

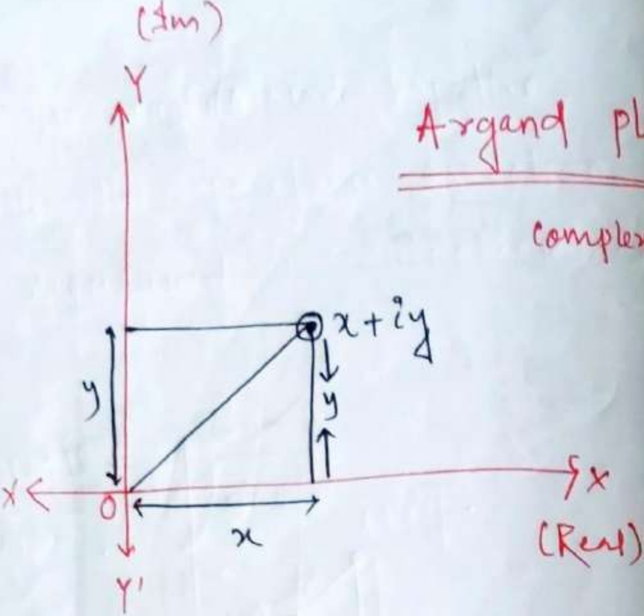
$$= \frac{3-4i}{9+16} = \frac{3-4i}{25} = 3/25 - 4/25\, i$$

Then $\bar{Z} = 3/25 + 4/25\, i$

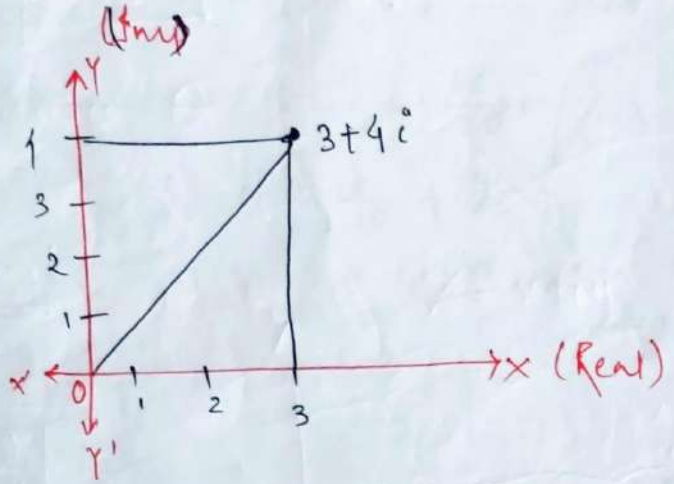
Geometrical Representation of a complex Number

Let $z = x + iy$

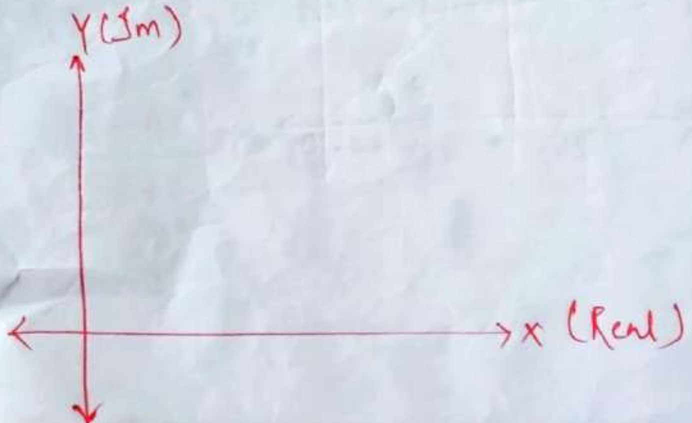
Argand plane / complex plane



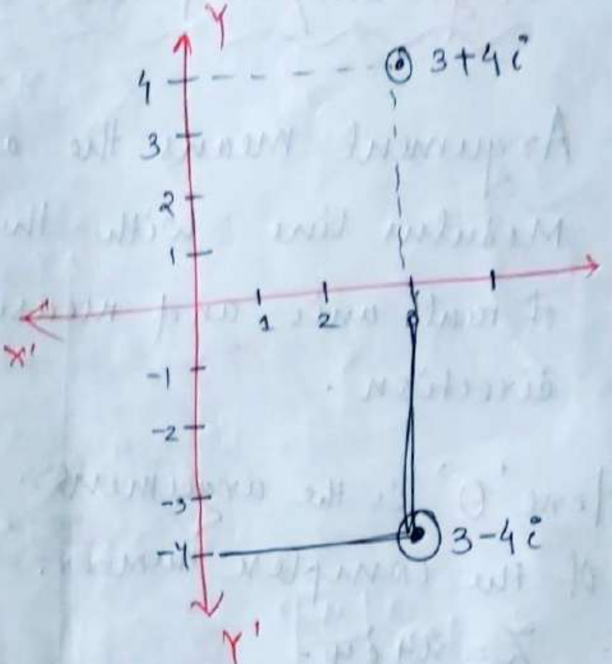
For example $z = 3 + 4i$



Example - 2 $z = 3 - 4i$



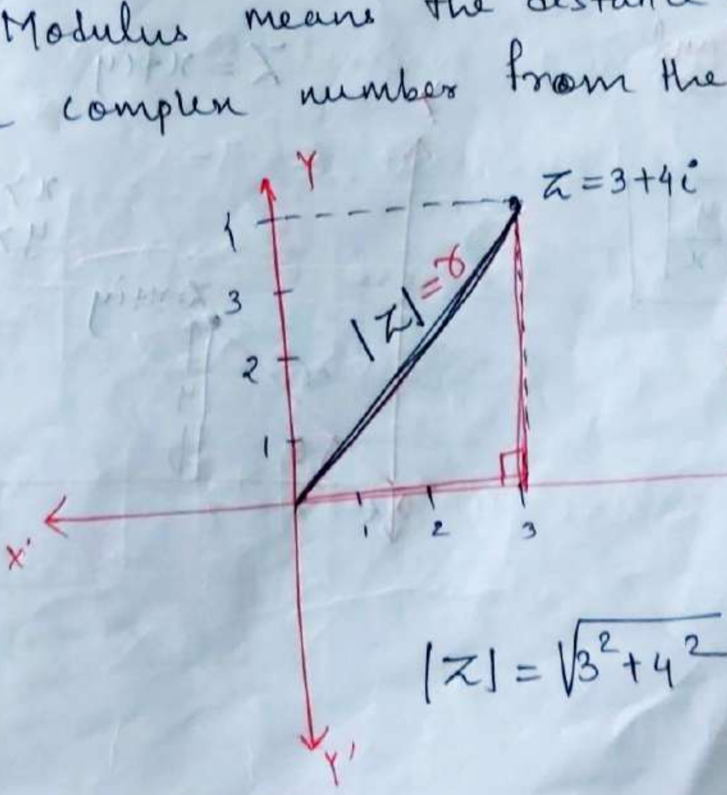
Example-2 $z = 3-4i$



NOTE conjugate of a complex Number is Mirror image of that no. across the real axis.

⇒ Modulus of $z = 3+4i$

NOTE Modulus means the distance of the given complex number from the origin.



using pythagoras theorem.

$$\boxed{h^2 = p^2 + b^2}$$

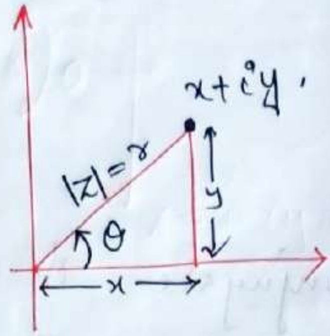
$$|z| = \sqrt{3^2+4^2} = \sqrt{25} = 5$$

Argument/Amplitude of a complex Number.

⇒ Argument means the angle(θ) made by the Modulus line with the positive direction of real axis and measured in anticlockwise direction.

⇒ Here 'θ' is the argument of the complex number.

$Z = x + iy$.



NOTE $\boxed{-\pi < \theta \leq \pi}$

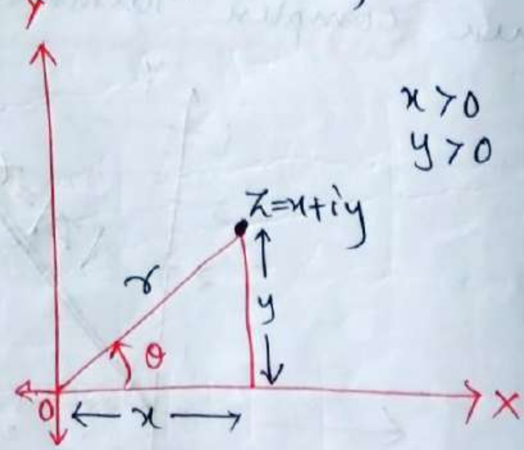
How to measure θ

For I^{st} quadant

$Z = x + iy$

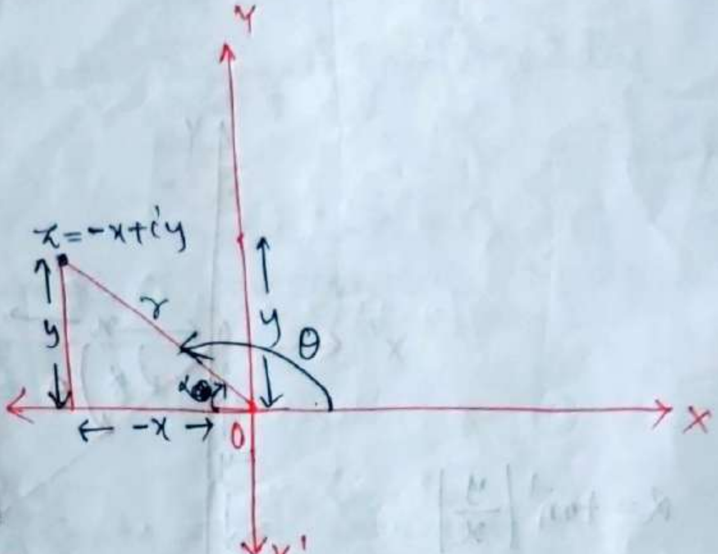
$$\boxed{\theta = \tan^{-1}\left|\frac{y}{x}\right|}$$

$x > 0$
$y > 0$



For 2^{nd} quadrant $(x<0, y>0)$

$z = -x + iy$

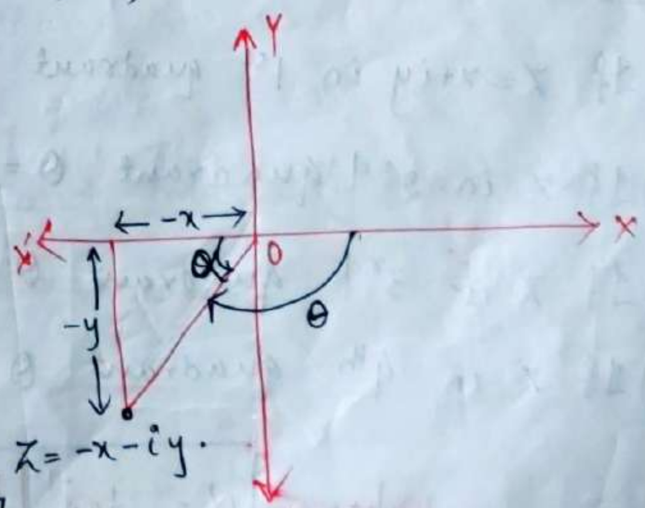


$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

Now $\boxed{\theta = \pi - \alpha}$

For 3^{rd} quadrant

$(x<0, y<0)$ $\quad z = -x - iy$



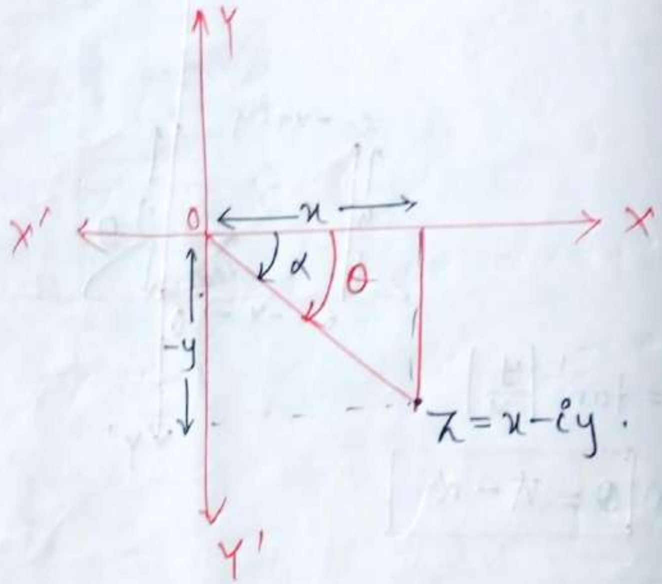
$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

Then $\boxed{\theta = -(\pi - \alpha)}$

$\boxed{\theta = \alpha - \pi}$

4^{th} quadrant $x>0, y<0$

$z = x - iy$



$\alpha = \tan^{-1}\left|\frac{y}{x}\right|$

$$\boxed{\theta = -\alpha}$$

NOTE.

① If $z = x+iy$ in 1^{st} quadrant $\theta = \alpha$

② If z in 2^{nd} quadrant $\theta = \pi - \alpha$

③ If z is 3^{rd} quadrant $\theta = -(\pi - \alpha)$

④ If z in 4^{th} quadrant $\theta = -\alpha$

where $\boxed{\alpha = \tan^{-1}\left|\frac{y}{x}\right|}$

CUBE ROOTS OF UNITY

Let cube root of unity is x.

i.e. $\sqrt[3]{1} = x$

$\Rightarrow 1 = x^3$

$\Rightarrow x^3 - 1 = 0$

$\Rightarrow (x-1)(x^2+x+1) = 0$

$\Rightarrow$ Either $x - 1 = 0$ or $x^2 + x + 1 = 0$

$\Rightarrow x = 1$ or $x = \dfrac{-1 \pm \sqrt{(1)^2 - 4\cdot 1\cdot 1}}{2\cdot 1}$

$\Rightarrow x = \dfrac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow x = \dfrac{-1 \pm i\sqrt{3}}{2}$

So the cube roots of unity are.

$$\boxed{1, \quad \frac{-1+i\sqrt{3}}{2} \text{ and } \frac{-1-i\sqrt{3}}{2}}$$

* One is real. other two roots are complex.

NOTE-1

Each complex cube root of unity is the square of the other.

i.e. $\left(\frac{-1+i\sqrt{3}}{2}\right)^2$

$$= \frac{(-1)^2 + (i\sqrt{3})^2 + 2(-1)(i\sqrt{3})}{4}$$

$$= \frac{1 + i^2(\sqrt{3})^2 - 2i\sqrt{3}}{4}$$

$$= \frac{1-3-2i\sqrt{3}}{4} = \frac{-2-2i\sqrt{3}}{4} = \frac{2(-1-i\sqrt{3})}{4}$$

$$= \frac{-1-i\sqrt{3}}{2}$$

* so if we take $\omega = \frac{-1+i\sqrt{3}}{2}$

then $\frac{-1-i\sqrt{3}}{2} = \omega^2$

NOTE-2

Sum of the cube roots of unity is zero.

i.e. $\boxed{1+\omega+\omega^2=0}$ or $\begin{cases} 1+\omega = -\omega^2 \\ 1+\omega^2 = -\omega \\ \omega+\omega^2 = -1 \end{cases}$

L.H.S $\quad 1 + \frac{-1+i\sqrt{3}}{2} + \frac{-1-i\sqrt{3}}{2}$

$$= \frac{2-1+\cancel{i\sqrt{3}}-1-\cancel{2\sqrt{3}}}{2}$$

$$= \frac{2-2}{2} = \frac{0}{2} = 0 \text{ (R.H.S)}$$

NOTE-3

As 'ω' is one of the cube root of unity

i.e. $\sqrt[3]{1} = \omega$

$\Rightarrow \boxed{1 = \omega^3}$

$\Rightarrow$ So we got $\omega^3 = 1$

Then $\omega^4 = \omega^3 \cdot \omega$
$= \omega$

$\omega^5 = \omega^3 \cdot \omega^2$
$= \omega^2$

$\omega^6 = (\omega^3)^2$
$= (1)^2 = 1$

$\Rightarrow$ $\omega^{56} = ?$

So $\omega^{56} = \omega^{3\times18+2}$

$= \omega^{3\times18} \cdot \omega^2$

$= (\omega^3)^{18} \cdot \omega^2$

$= 1 \cdot \omega^2 = \omega^2$

$$\begin{array}{r|l|l} 3 & 56 & 18 \\ & 3 & \\ \hline & 26 & \\ & 24 & \\ \hline & 2 & \end{array}$$

$56 = 3\times18+2$

De-Moivre's Theorem

Theorem – If n is an integer, positive or negative or zero, Then

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta).$$

NOTE-1 $(\cos\theta + i\sin\theta)^{-n} = \cos(-n)\theta + i\sin(-n)\theta$

$= \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

NOTE-2 $(\cos\theta + i\sin\theta)^0 = \cos(0\cdot\theta) + i\sin(0\cdot\theta)$

$= \cos 0 + i\sin 0$

$= 1 + i\cdot 0 = 1$

Q:- find the value of $(\cos 60^\circ + i\sin 60^\circ)^3$

Ans:- $(\cos 60^\circ + i\sin 60^\circ)^3$

$= \cos(3\times 60^\circ) + i\sin(3\times 60^\circ)$

$= \cos 180^\circ + i\sin 180^\circ$

$= -1 + i\cdot 0$

$= -1$

Square root of a complex Number :-

Q.1 Find the Square roots of $3+4i$

Sol^n Let $\sqrt{3+4i} = x+iy$

$\Rightarrow 3+4i = (x+iy)^2$

$\Rightarrow 3+4i = x^2 + 2(x)(iy) + (iy)^2$

$\Rightarrow 3+4i = x^2 + 2ixy + i^2y^2$

$\Rightarrow 3+4i = x^2 + 2ixy - y^2 \quad (\because i^2 = -1)$

$\Rightarrow 3+4i = x^2 - y^2 + i\,2xy$

Equating real and imaginary parts.

$x^2 - y^2 = 3$

and $2xy = 4$

Now consider. $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$(\because \cancel{(a^2+b^2)^2 =})$

$\Rightarrow (x^2+y^2)^2 = (3)^2 + (2xy)^2 \qquad \{\because (a+b)^2 = (a-b)^2 + 4ab\}$

$= 9 + (4)^2 = 9+16$

$= 25$

$\Rightarrow x^2+y^2 = \sqrt{25} = \pm 5 = 5$

Solving

Now, $x^2+y^2=5$

$x^2-y^2=3$

$2x^2=8$

$x^2=4$

$\Rightarrow x=\pm 2$

putting value of x in $x^2+y^2=5$

$\Rightarrow (\pm 2)^2+y^2=5$

$\Rightarrow 4+y^2=5$

$\Rightarrow y^2=1$

$\Rightarrow y=\sqrt{1}=\pm 1$

Hence Square roots of $3+4i = \pm 2 \pm 1i$

$= \pm (2+i)$

CHAPTER-2 MATRIX

Definition :-

Matrix is a rectangular array of numbers arranged in rows and columns.

$$\text{e.g. } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

order

⇒ If there are 'm' no. of rows and 'n' no. of columns, then the matrix is of order $m \times n$

$$\text{e.g. } A = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 0 & 5 \end{bmatrix}_{2\times 3}$$

order $= 2\times 3$

2 = no. of rows

3 = no. of columns.

⇒ Sub Matrix

Any Matrix obtained by omitting some rows are columns or both of a given $m\times n$ Matrix is called its sub Matrix.

e.g. $A = \begin{bmatrix} 2 & 0 & 4 \\ 5 & 2 & 1 \\ 7 & -1 & 4 \end{bmatrix}$ Then $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ is one of its Sub-Matrix

Determinant of a Matrix

⇒ Let A is a square Matrix of order 2

i.e. $A = \begin{bmatrix} 4 & 5 \\ 1 & -6 \end{bmatrix}$

Then Determinant of A is obtained by

$$|A| = \begin{vmatrix} 4 & 5 \\ 1 & -6 \end{vmatrix} = -24 - 5$$
$$= -29$$

⇒ Let A is a Square Matrix of order 3.

i.e. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix}$$

$$= 1\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix} - 2\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} + 3\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= 1(28-30) - 2(21-25) + 3(18-20)$$

$$= 1(-2) - 2(-4) + 3(-2)$$

$$= -2 + 8 - 6 = 0$$

Rank of a Matrix

The rank is how many rows (or columns) of the Matrix are unique (linearly independent).

* Rank of a Matrix A is denoted by $r(A)$

e.g rank of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}$

Solⁿ let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix} \begin{matrix} \to R_1 \\ \to R_2 \end{matrix}$

Here $R_2 \to 3R_1$

So, here only R_1 is unique (linearly independent)

So $r(A) = 1$

Example 2 $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 4 & 5 \end{bmatrix}$

Here R_1 is unique

R_2 is unique (not dependent on R_1)

but $R_3 \to R_1 + R_2$ (i.e. dependent)

So $r(A) = 2$

Example 3 $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & -2 & -1 \end{bmatrix}$

Here R_1 is unique.

R_2 is also linearly independent.

$R_3 \to R_1 - 2R_2$ (linearly dependent).

$\Rightarrow r(A) = 2$

Another Method using determinant · rank

A number 'r' is said to be rank of a non-zero matrix if

* There exist atleast one square submatrix of order 'r' whose determinant is not equal to zero.

* And determinant of every square matrix of order $(r+1)$ is zero.

Q:- 1 find the rank of $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solⁿ Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

* As A is a Matrix of order 3×3

$r(A) \le 3$

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= 0 \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}$$

$$= +1(0-2)$$

$$= -2 \neq 0$$

So rank of A or $r(A) = 3$

Q.2 Find the rank of the Matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solⁿ Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\qquad r(A) \leq 3$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(2-1) - 2(3-1) + 3(3-2)$$

$$= 1 - 2(2) + 3(1)$$

$$= 1 - 4 + 3$$

$$= 0$$

So $r(A) \neq 3$

Then consider any 2nd order Sub-Matrix.

$$\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = 2 - 6 = -4 \neq 0$$

So $r(A) = 2$

Another Method

Elementary Transformation

In elementary Transformation a Matrix is obtained by using(applying) a Series of elementary row operations s.t. there are some non-zero rows on the top and the remaining rows consist of all zeros is called canonical Matrix.

$\Rightarrow$ Then the rank is the number of non-zero rows.

Elementary Operations

(1) Interchange of any two rows or columns.

for example
$$\begin{bmatrix} 3 & 0 & 2 \\ 4 & 2 & 1 \\ 5 & -2 & 5 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 2 \\ 5 & -2 & 5 \end{bmatrix}$$

(2) Multiplication of any row by a non-zero number 'k'.

$$\begin{bmatrix} 4 & 2 \\ -5 & 4 \end{bmatrix}$$

$R_2 \to 3R_2$

$$= \begin{bmatrix} 4 & 2 \\ -15 & 12 \end{bmatrix}$$

(3) ~~Multipt~~ Adding a row by Multiplying a non zero number 'k' to another row.

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 0 & 2 \\ 5 & -1 & 4 \end{bmatrix}$$

$R_2 \to R_2 + 2R_3$

$$\begin{array}{rrr} & 4 & 0 & 2 \\ + & 10 & -2 & 8 \\ \hline & 14 & -2 & 10 \end{array}$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 14 & -2 & 10 \\ 5 & -1 & 4 \end{bmatrix}$$

Q:- Find the rank of the followings.

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$

(*) V.V.Imp. NOTE Make below elements of diagonal as zero i.e. Numbers that are rounded.
→ After that make other elements zero if possible without hampering other zero numbers

Sol^n

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$

$\approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\approx \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

$R_2 = 3\ 4\ 5$

$3R_1 = 3(1\ 2\ 3)$

$= 3\ 6\ 9$

$R_2 - 3R_1$

$\begin{array}{r} 3\ 4\ 5 \\ -\ 3\ 6\ 9 \\ \hline 0\ -2\ -4 \end{array}$ (Ans)

no. of non zero rows = 2

$r(A) = 2$

⇒ Rank of a Matrix is used to find solution of linear system of eqⁿs.

Solution of linear system of Eqⁿs :- (Non-homogeneous)

(i) So, consider the set of 'm' ~~linear eqⁿs~~ non-homogeneous linear equations.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = K_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = K_2 \\ - \quad - \quad - \quad - \quad - \quad - \\ - \quad - \quad - \quad - \quad - \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = K_m \end{array}\right\} \text{— (i)}$$

The above system of eqⁿs can be written in ~~con~~ Matrix form i.e.

$$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_m \end{bmatrix}$$

here A is called co-efficient Matrix.

and $K = \left[\begin{array}{cccc|c} a_{11} & \cdots & \cdots & a_{1n} & k_1 \\ a_{21} & \cdots & \cdots & a_{2n} & k_2 \\ a_{31} & \cdots & \cdots & a_{3n} & k_3 \\ - & - & - & - & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} & k_m \end{array}\right]$

is called augmented Matrix.

NOTE 1:- The system of linear eqⁿs is said to be consistent if and only if the coefficient Matrix 'A' and the augmented Matrix 'K' are of same rank otherwise the system is inconsistent.
And the above statement is known as

Rouche's Theorem.

NOTE 2:- Let rank of $A = r$, rank of $K = r'$. And n = no. of unknowns

Linear system of eqⁿs

- (consistent (or) solution $(r = r')$
  - $r = r' = n$ (unique solⁿ)
  - $(r = r' < n)$ (infinitely many solⁿ)
- No solution $(r \neq r')$ or inconsistent.

(ii) Solution of linear system of eq^ns (Homogeneous)

Consider the 'm' number of homogeneous linear equations.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$- - - - - - - - -$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

The above ~~Matrix~~ equations can be written in Matrix form.

$$AX = B$$

Where $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & - - & a_{2n} \\ - & - - & \cdots \\ a_{m1} & - - & a_{mn} \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

As 'B' is a Zero Matrix.

So your Augmented Matrix (K) is equal to 'A'

NOTE :-

(1) If rank of $A = n$ (no. of unknowns) after elementary row operation.

then the given linear eq^n's have only a trivial solution.

i.e. $x_1 = x_2 = \cdots = x_n = 0$

(2) If rank of $A < n$

then eq^n's have infinitely many solutio

Q.1 solve $x - y + z = 0$

$x + 2y - z = 0$

$2x + y + 3z = 0$

Solution The given linar eq^n's can be written

in $AX = 0$

where $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Consider $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

$r(A) = 3 =$ no. of unknowns.

Hence the eq^{n}s have only trivial solution

i.e. $x = y = z = 0$

Linear Differential Equations :-

Definition :-

Differential Equation in which the dependent Variable (y) and all its derivatives are of degree '1' and are not multiplied together, is called a linear differential equation.

(i) linear differential equation of order 1 :-

Linear differential equation of 1st order is given by

$$\boxed{\frac{dy}{dx} + Py = Q}$$

where P and Q either functions of 'x' or constants.

(ii) Linear differential equation of n^{th} order :-

Linear differential equation of n^{th} order is given by

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = X$$

where $P_1, P_2, P_3 \ldots P_n$ and X either functions of 'x' or constants.

Linear Differential equation of nth order with constant coefficients :-

L.D.E of n^{th} order with constant coefficients is given by

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + K_n y = X \quad \text{(I)}$$

where the coefficients $K_1, K_2, \cdots K_n$ are constants.

* Now by using Operator 'D' eq^n (1) can be written as.

$$D^n y + K_1 D^{n-1} y + K_2 D^{n-2} y + \cdots + K_n y = X \quad \text{(II)}$$

where $\boxed{D = \frac{d}{dx}}$ $\boxed{D^2 = \frac{d^2}{dx^2}}$ $\Rightarrow \frac{d^2y}{dx^2} = D^2 y$

Then $\frac{dy}{dx} = Dy$ and so -- on...

Now eq^n (II) $\Rightarrow \left(D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n\right) y = X$

$$\Rightarrow f(D)\, y = X \quad \text{(III)}$$

where $f(D)$ represents a polynomial in D.

Solution of L·D·E of nth order with constant coefficients:-

A complete solution of

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + K_n y = X$$

is given by $y = u + v$

where 'u' is called complementary function (C·F.) & is obtained both for homogeneous L·D·E & non-homogeneous L·D·E.

And 'v' is called particular Integral (P·I·) which is obtained for only non-homogeneous L·D·E.

NOTE:-

(1) If $D^n y + K_1 D^{n-1} y + \cdots + K_n y = 0$, it is called a homogeneous L·D·E of nth order.

(2) If $D^n y + K_1 D^{n-1} y + \cdots + K_n y = X$, it is called a non-homogeneous L·D·E of nth order.

(3) For homogeneous L·D·E i.e. $f(D)y = 0$, sol^n is $y = C.F.$ only.

(4) for non-homogeneous L.D.E. i.e. $f(D)y = X$

Sol^n is $\rightarrow y = C.F + P.I.$

<u>Rules for finding the complementary function:-</u>

Given L.D.E is

$$\frac{d^ny}{dx^n} + K_1 \frac{d^{n-1}y}{dx^{n-1}} + K_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots\cdots + K_n y = 0 \quad \text{——— (I)}$$

(i.e. a homogeneous L.D.E).

$\Rightarrow D^n y + K_1 \underline{D^{n-1}y} + K_2 D^{n-2} y + \cdots K_n y = 0$

(by using Operator D)

$\Rightarrow (D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n) y = 0$

$\Rightarrow f(D) y = 0 \quad \text{——— (II)}$

* Then by equating the symbolic coefficients to zero

i.e. $D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n = 0$

we will get the A.E. of ~~$f(D)y = 0$~~ . eq^n (II)

i.e. $f(D) = 0$ is the Auxillary ~~eq~~ equation of order 'n'

So by solving $f(D) = 0$ ——— (III)

we will get 'n' no. of roots.

i.e. let $m_1, m_2, \cdots m_n$.

Then there ~~ants~~ arises foure cases depending upon the roots.

Case I :- If the roots are real and unequal.

If the roots will different

Then eqn (II) i.e. $f(D)y = 0$ becomes.

$\Rightarrow (D - m_1)(D - m_2) \cdots (D - m_n)y = 0$

Now taking $(D - m_1)y = 0$

$\Rightarrow Dy - m_1 y = 0$

$\Rightarrow \frac{dy}{dx} - m_1 y = 0$ which is of the form

$$\boxed{\frac{dy}{dx} + Py = Q}$$

then its soln is obtained as follows:

Soln

given $\frac{dy}{dx} - m_1 y = 0$ here $P = -m_1$

$Q = 0$

then $I.f = e^{\int P\,dx}$

$= e^{\int -m_1\,dx}$

$= e^{-m_1 x}$

then soln is given by

$y \times I.f = \int (Q \times I.f)\,dx$

$\Rightarrow y\,e^{-m_1 x} = \int 0\,dx$

$\Rightarrow y\,e^{-m_1 x} = c_1$

$\Rightarrow y = c_1 e^{m_1 x}$

Then for the factor $(D - m_2)y = 0$

$y = c_2 e^{m_2 x}$

for $(D - m_3)y = 0$

$\Rightarrow y = c_3 e^{m_3 x}$ and so -- on ---

Then the complete sol^n of $f(D)y = 0$

is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$

NOTE :- If the roots are real & unequal.

Then

$$\boxed{C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}}$$

Where $m_1, m_2 \cdots m_n$ are the roots.

Case II :- If the roots are real and equal.

Let $m_1, m_2 \cdots m_n$ are the real roots.

(I) Where $m_1 = m_2$ and $m_3, m_4 \cdots m_n$ are different

then $$\boxed{C.F = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}}$$

(II) If $m_1 = m_2 = m_3$ and $m_4, m_5 \cdots m_n$ are different.

then $$\boxed{C.F = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_4 x}}$$

Case III :- If Roots are complex and unequal.

let $m_1 = \alpha + i\beta$

$m_2 = \alpha - i\beta$ are two complex roots

and $m_3, m_4 - - - m_n$ are real & different roots.

Then

$$\boxed{C.F = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}}$$

Case-IV :- If the roots are complex and equal.

Let $m_1 = m_2 = \alpha + i\beta$

$m_3 = m_4 = \alpha - i\beta$

and $m_5, m_6 - - -, m_n$ are real & different roots.

Then.

$$\boxed{C.F = e^{\alpha x}\left[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x\right] + c_5 e^{m_5 x} + \cdots}$$

Q.1 Solve $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

Solution Given L.D.E is

$$\Rightarrow \frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0 \quad \text{——(I)}$$

using Operator D

$$\Rightarrow D^2y + 7Dy + 12y = 0$$

$$\Rightarrow (D^2 + 7D + 12)y = 0 \quad \text{——(II)}$$

Auxilliary Equation of eq^n (II) is given by

$$\Rightarrow D^2 + 7D + 12 = 0$$

Then $D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-7 \pm \sqrt{(7)^2 - 4 \cdot 1(12)}}{2(1)}$$

$$= \frac{-7 \pm 1}{2}$$

So $D = \frac{-7+1}{2}$ or $\frac{-7-1}{2}$

$= -3$ or -4

So here the roots -3 & -4 are real and diff.

So $C.F = c_1 e^{-3x} + c_2 e^{-4x}$ which

is the compute solution of eq^n(1).

Q.2 solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

Solution Given $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ ——(1)

$\Rightarrow D^2y - 2Dy + y = 0$ [using Operator D]

$\Rightarrow (D^2 - 2D + 1)y = 0$ ——(II)

Auxillary eq^n is given by

$\Rightarrow D^2 - 2D + 1 = 0$ ——(III)

Solving eq^n (III) $D = \frac{2 \pm \sqrt{4-4}}{2}$

$= \frac{2}{2}$

$D = 1$ or 1 (as $D^2 - 2D + 1$ is a quadratic eq^n, so we will get 2 roots)

Here the roots 1 & 1 are real & equal

So $C.F = (c_1 x + c_2) e^x$ is the

complete soln of eqn (1).

Q.3 Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Solution Given L.D.E is $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$ —(I)

$\Rightarrow D^2y + 4Dy + 13y = 0$ (using Operator D)

$\Rightarrow (D^2 + 4D + 13)y = 0$ —(II)

A.E. is given by

$D^2 + 4D + 13 = 0$ —(III)

Now solving (III) i.e. $D = \frac{-4 \pm \sqrt{16-52}}{2}$

$= \frac{-4 \pm \sqrt{-36}}{2}$

$= \frac{-4 \pm i6}{2}$

So $D = \frac{-4+6i}{2}$ or $\frac{-4-6i}{2}$

$\sqrt{36} = 6$

but $\sqrt{-36}$

$= \sqrt{(-1)36}$

$= \sqrt{i^2 36}$

$= i6$ as $\boxed{-1 = i^2}$

$\Rightarrow D = -2+3i$ or $-2-3i$

here $\alpha = -2$

$\beta = 3$

then $C.F = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$= e^{-2x}(C_1 \cos 3x + c_2 \sin 3x)$

Which is the complete solution.

Q.4 solve $\frac{d^3y}{dx^3} + y = 0$

Solution :- Given $\frac{d^3y}{dx^3} + y = 0$ —— (I)

$\Rightarrow D^3y + y = 0$ (using Operator D)

$\Rightarrow (D^3+1)y = 0$ —— (II)

Then A.E. is given by

$D^3+1=0$

Solving $D^3+1=0$

$\Rightarrow (D)^3 + (1)^3 = 0$

$\Rightarrow (D+1)(D^2 - D + 1) = 0$

If $D+1=0 \Rightarrow \boxed{D=-1}$

if $D^2-D+1=0$

$$\Rightarrow D = \frac{1 \pm \sqrt{1-4(1)(1)}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$= \frac{1}{2} \pm i\sqrt{3}/2$$

So $\boxed{D = \frac{1}{2} + i\frac{\sqrt{3}}{2}}$ & $D = \boxed{\frac{1}{2} - i\frac{\sqrt{3}}{2}}$

Here $\underbrace{-1}_{\text{real root.}}$, $\underbrace{\frac{1}{2} + i\frac{\sqrt{3}}{2} \text{ \& } \frac{1}{2} - i\frac{\sqrt{3}}{2}}_{\text{complex roots}}$ are the roots

then $C.F = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2} + c_2 \sin\frac{\sqrt{3}}{2}\right)$

is the complete solution.

Rules to find Particular Integral (P.I)

To find P.I of L.D.E we need an Inverse differential operator.

Inverse differential Operator :-

we already have discussed differential operator that is $D = \frac{d}{dx}$.

And the Inverse Operator is denoted by $\frac{1}{D}$ which stands for Integral.

Now consider the eqn.

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1}y}{dx^{n-1}} + K_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + K_n y = X \quad \text{(1)}$$

$$\Rightarrow D^n y + K_1 D^{n-1} y + K_2 D^{n-2} y + \cdots + K_n y = X$$

(using differential operator)

$$\Rightarrow (D^n + K_1 D^{n-1} + K_2 D^{n-2} + \cdots + K_n) y = X$$

~~then~~ $\Rightarrow f(D) y = X$

$\Rightarrow y = \frac{1}{f(D)} X$, which is the P.I. of eqn (1).

Then there arises four cases depending upon the 'X'

Case I when $X = e^{ax}$ (i.e. an exponential function)

Then $P.I = \frac{1}{f(D)} e^{ax}$.

$= \frac{1}{f(a)} e^{ax}$, provided $f(a) \neq 0$.

If $f(a) = 0$

Then $P.I = \frac{1}{f(D)} e^{ax}$

$= \frac{x}{f'(a)} e^{ax}$, provided $f'(a) \neq 0$.

If $f'(a) = 0$ Then $P.I = \frac{x^2}{f''(a)} e^{ax}$, provided $f''(a) \neq 0$

and so...on.

Q.1 find P.I. of $(D^2+5D+6)y = e^x$

Solution :- Given $(D^2+5D+6)y = e^x$

$$\Rightarrow P.I = \frac{1}{D^2+5D+6} e^x.$$

put $D=1$

$$\Rightarrow P.I = \frac{1}{12} e^x.$$

(Ans).

Q.2 find P.I. of $(D-2)^2 y = e^{2x}$.

Solution :- Given $(D-2)^2 y = e^{2x}$

$$\Rightarrow P.I = \frac{1}{(D-2)^2} e^{2x}$$

$$= \frac{1}{D^2-4D+4} e^{2x}$$

Putting $D=2$ $f(D)=0$, so case failure.

~~so find $f'(D)$~~.

$$= \frac{x}{2D-4} e^{2x}$$

Putting $D=2$ $f'(D)=0$, so case fail

$$= \frac{x^2}{2} e^{2x} \quad \text{(Ans)}.$$

Case II :- If $X = \sin(ax+b)$ or $\cos(ax+b)$.

Then $P.I = \frac{1}{f(D^2)} \sin(ax+b)$

$$= \frac{1}{f(-a^2)} \sin(ax+b), \text{ provided } f(-a^2) \neq 0.$$

If $f(-a^2) = 0$, then rule fails.

$$P.I = \frac{x}{f'(D^2)} \sin(ax+b).$$

$$= \frac{x}{f'(-a^2)} \sin(ax+b), \text{ provided } f'(-a^2) \neq 0$$

and so ... on.

Q:- find P.I. of $(D^2+D+1)y = \sin 2x$.

Solution :- $P.I = \frac{1}{D^2+D+1} \sin 2x$.

put $-a^2$ in place of D^2

$$= \frac{1}{-4+D+1} \sin 2x.$$

$$= \frac{1}{D-3} \sin 2x.$$

$$= \frac{(D+3)}{(D-3)(D+3)} \sin 2x$$

$$= \frac{D+3}{D^2-9} \sin 2x$$

put $-a^2$ in place of D^2

$$= \frac{D+3}{-4-9} \sin 2x$$

$$= \frac{D+3}{-13} \sin 2x$$

$$= -\frac{1}{13} [D \sin 2x + 3 \sin 2x]$$

$$= -\frac{1}{13} [2 \cos 2x + 3 \sin 2x]$$ (Ans)

Q.2 find the P.I. of $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$.

Soln Given eqn is $D^3y + 4Dy = \sin 2x$

$\Rightarrow (D^3 + 4D)y = \sin 2x$.

$$P.I = \frac{1}{D^3 + 4D} \sin 2x$$

$= \dfrac{1}{D^2 \cdot D + 4D} \sin 2x$.

\# Put $-a^2$ in place of D^2

$= \cancel{\dfrac{1}{-4D + 4D}} \cancel{x}$ case failure,

$= \dfrac{x}{3D^2 + 4} \sin 2x$.

put $-a^2$ in place of D^2

$= \dfrac{x}{3(-4) + 4} \sin 2x$.

$= \dfrac{x}{-8} \sin 2x$ (Ans).

Case III :- when $X = x^m$

$P.I = \dfrac{1}{f(D)} x^m$.

$= [f(D)]^{-1} x^m$

Here we are going to use two binomial expansion

(i) $(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$

(ii) $(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$

Q.2 find P.I. of $(D^2+3D+2)y = x^2$

Sol^n

$$P.I. = \frac{1}{D^2+3D+2}x^2$$

$$= \left[D^2+3D+2\right]^{-1}x^2$$

$$= \left[2\left(1+\frac{D^2+3D}{2}\right)\right]^{-1}x^2$$

$$= \frac{1}{2}\left[1+\frac{D^2+3D}{2}\right]^{-1}x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2+3D}{2}+\left(\frac{D^2+3D}{2}\right)^2 - \cdots\cdots\right]x^2$$

$$= \frac{1}{2}\left[x^2 - \frac{D^2+3D}{2}(x^2) + \frac{(D^4+9D^2+6D^3)}{2}x^2\right]$$

$$= \frac{1}{2}\left[x^2 - \left(\frac{2+6x}{2}\right) + \left(\frac{0+18+0}{2}\right)\right]$$

$$= \frac{1}{2}\left[x^2 - (1+3x) + 9\right]$$

$$= \frac{1}{2}\left[x^2 - 3x + 8\right] \text{ (Ans).}$$

Case-IV :- when $X = e^{ax} V$, where V being a function of x.

Then $P.I = \frac{1}{f(D)} e^{ax} V$

$$= e^{ax} \frac{1}{f(D+a)} V.$$

Q.1 find P.I. of $(D^2-2D+4)y = e^x \cos x$.

Solution :- $P.I = \frac{1}{D^2-2D+4} e^x \cos x$.

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x.$$

$$= e^x \frac{1}{D^2+2D+1-2D-2+4} \cos x,$$

$$= e^x \frac{1}{D^2+3} \cos x.$$

$= \cancel{e^x}$ put $D^2 = -a^2 = -1$

$$= e^x \frac{1}{-1+3} \cos x.$$

$$= \frac{e^x}{2} \cos x \quad \text{(Ans)}$$

Q.2 find P.I. of $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=e^{-2x}\sin 2x$.

Sol^n Given $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=e^{-2x}\sin 2x$.

$$\Rightarrow D^2y+5Dy+6y=e^{-2x}\sin 2x.$$

$$\Rightarrow (D^2+5D+6)y=e^{-2x}\sin 2x.$$

$$P.I=\frac{1}{D^2+5D+6}e^{-2x}\sin 2x$$

$$=e^{-2x}\frac{1}{(D-2)^2+5(D-2)+6}\sin 2x.$$

$$=e^{-2x}\frac{1}{D^2-4D+4+5D-10+6}\sin 2x.$$

$$=e^{-2x}\frac{1}{D^2+D}\sin 2x.$$

put $D^2=-a^2=-4$

$$=e^{-2x}\frac{1}{-4+D}\sin 2x.$$

$$=e^{-2x}\frac{D+4}{(D-4)(D+4)}\sin 2x.$$

$$=e^{-2x}\frac{D+4}{D^2-16}\sin 2x.$$

$$= e^{-2x}\frac{(D+4)}{-4-16}\sin 2x.$$

$$= \cancel{\frac{e^{-2x}}{-20}\sin 2x} = \frac{e^{-2x}}{-20}(D\sin 2x + 4\sin 2x)$$

~~(Ans)~~

$$= \frac{e^{-2x}}{-20}(2\cos 2x + 4\sin 2x)$$

(Ans).

Q.1 Solve

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x$$

Sol^n Given diff eq^n is

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x.$$

$\Rightarrow D^3y + 2D^2y + Dy = e^{2x} + \sin 2x$ (using diff. operator D)

$\Rightarrow (D^3 + 2D^2 + D)\, y = e^{2x} + \sin 2x.$

Then A.E is given by.

$$D^3 + 2D^2 + D = 0$$

$$\Rightarrow D(D^2 + 2D + 1) = 0$$

So either $D = 0$ or $D^2 + 2D + 1 = 0$

$\Rightarrow D = -1, -1$

So the roots are $D = 0, -1, -1$.

Then $C.F = C_1 e^{0.x} + (C_2 x + C_3)\, e^{-1(x)}$

$= C_1 + (C_2 x + C_3)\, e^{-x}.$

$$\text{And } P.I = \frac{1}{D^3+2D^2+D}\left(e^{2x}+\sin 2x\right)$$

$$= \underbrace{\frac{1}{D^3+2D^2+D}e^{2x}}_{P.I_1} + \underbrace{\frac{1}{D^3+2D^2+D}\sin 2x}_{P.I_2}$$

$$P.I_1 = \frac{1}{D^3+2D^2+D}e^{2x}$$

$$= \frac{1}{(2)^3+2(2)^2+2}e^{2x}$$

$$= \frac{e^{2x}}{18}$$

$$P.I_2 = \frac{1}{D^3+2D^2+D}\sin 2x.$$

$$= \frac{1}{D^2\cdot D+2D^2+D}\sin 2x,$$

Put $D^2 = -a^2 = -4$

$$= \frac{1}{-4D+(-8)+D}\sin 2x.$$

$$= \frac{1}{-3D-8} \sin 2x.$$

$$= \frac{-3D+8}{(-3D-8)(-3D+8)} \sin 2x$$

$$= \frac{-3D+8}{(-3D)^2-(8)^2} \sin 2x.$$

$$= \frac{-3D+8}{9D^2-64} \sin 2x.$$

put $D^2 = -4$

$$= \frac{-3D+8}{-36-64} \sin 2x.$$

$$= \frac{-3D+8}{-100} \sin 2x.$$

$$= \frac{-1}{100} \left(-3D \sin 2x + 8 \sin 2x\right)$$

$$= \frac{-1}{100} \left(-3(2\cos 2x) + 8 \sin 2x\right).$$

Linear Partial Differential Equation of first order (Lagrange's form).

Lagrange's linear equation is of the form

$$\boxed{Pp + Qq = R} \quad \text{—(i) (standard form)}$$

where P, Q & R are functions of x, y, z and this is called a quasi-linear eqn.

and $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$

Solution of linear P.D.E of first order.

⇒ first find A.E for eqn (i)

i.e. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ —— (ii)

⇒ and we have to solve the above A.E.

and this can be done by using two Methods.

(i) Grouping Method.

(ii) Method of Multipliers,

(iii) Combination of (i) & (ii)

⇒ Suppose $U = a$ and $V = b$ are two solⁿs of A·E by using any of the Methods where U and V are functions of x, y, z.

⇒ Then complete solution of eqⁿ (1) is

$$f(a, b) = 0$$

$$\text{or } f(u, v) = 0$$

or $u = \phi(v)$ or $v = \phi(u)$.

(1) Grouping Method :-

we have to make groups from eqⁿ (11) which can be easily integrable.

like make $\frac{dx}{P} = \frac{dy}{Q}$ (one group)

and by solving the differential eqⁿ.

we get $u = a$ (one solution)

again we can make $\frac{dy}{Q} = \frac{dz}{R}$ (another group)

and by solving above

we will get $u = b$ (2nd solution)

(II) Method of Multipliers :-

In this Method choose Multipliers l, m, n Such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{Pl + Qm + Rn}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

Choose l, m, n s.t. the ~~fourth~~ denominator of fourth Part of A.E will be zero.

& $l\,dx + m\,dy + n\,dz$ will be integrate easily.

Let's make a group,

$$\Rightarrow \frac{dx}{P} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

$$\Rightarrow l\,dx + m\,dy + n\,dz = 0 \quad \text{——— (1)}$$

And Integrating the eq^n (1) we will get a sol^n $u = a$

→ Similarly choose another set of multipliers s.t. we will get another sol^n $v = b$.

Then complete sol^n is $f(u, v) = 0$

(III) Combination of (I) & (II) :-

In some cases we will get a sol^n from grouping Method and another sol^n is obtained from Method of Multipliers.

Q.1 Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Sol^n Given P.D.E $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ — (1)
(Which is a Partial L.D.E in Lagrange's form).

$\Rightarrow$ Then A.E. of (1) is given by

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}} \quad \text{— (II)} \quad \left(\begin{array}{l} \text{as } P = \sqrt{x} \\ Q = \sqrt{y} \\ R = \sqrt{z} \end{array}\right)$$

Considering first two ratios, and make a group.

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$$

$$\Rightarrow \int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}}$$

$$\Rightarrow \int x^{-1/2}\, dx = \int y^{-1/2}\, dy.$$

$$\Rightarrow 2\sqrt{x} = 2\sqrt{y} + 2c_1$$

$$\Rightarrow \sqrt{x} - \sqrt{y} = c_1 \text{ — (I)}$$

Again considering last two ratios and make a group

$$\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

$$\Rightarrow \int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$$

$$\Rightarrow 2\sqrt{y} = 2\sqrt{z} + 2c_2$$

$$\Rightarrow \sqrt{y} - \sqrt{z} = c_2 \text{ — (II)}$$

So the required sol^n.

$$\phi(\sqrt{x}-\sqrt{y},\ \sqrt{y}-\sqrt{z})=0$$

or $\sqrt{x}-\sqrt{y}=f(\sqrt{y}-\sqrt{z})$

Q.2 Solve $\frac{y^2 z}{x} p + xz q = y^2$

Sol^n $y^2 z p + x^2 z q = y^2 x$ — (1)

Given Partial L.D.E is in Lagrange's form.

Here $P = y^2 z$, $Q = x^2 z$ and $R = y^2 x$.

Then A.E for (1) is given by $\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$ —

Take first two ratios and make a group.

$$\frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}}$$

$\Rightarrow x^2 dx = y^2 dy$

Integrate both sides.

$\Rightarrow \int x^2 dx = \int y^2 dy$

$\Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + c_1$

$\Rightarrow \frac{x^3}{3} - \frac{y^3}{3} = c_1$

$\Rightarrow x^3 - y^3 = c_1$ — (iii)

Again take first and third ratio to make another group

$$\frac{dx}{y^2 z} = \frac{dz}{y^2 x}$$

$\Rightarrow \frac{dx}{z} = \frac{dz}{x}$

$\Rightarrow x dx = z dz$

Integrate both sides.

$\Rightarrow \int x dx = \int z dz$

$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + c$

$\Rightarrow \frac{x^2}{2} - \frac{z^2}{2} = c$

$\Rightarrow x^2 - z^2 = 2c$

$\Rightarrow x^2 - z^2 = c_2$ — (iv)

So the complete sol^n is $\phi(x^3 - y^3,\ x^2 - z^2) = 0$

Q.1 Solve $x(y-z)p + y(z-x)q = z(x-y)$

Solution Here $P = x(y-z)$

$$Q = y(z-x)$$

$$R = z(x-y)$$

Then auxilliary Eq^n is given by

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \quad \text{——— (1)}$$

We can solve this A.E. by using Method of multipliers.

Choose ~~to~~ multiplies $l=1$, $m=1$ & $n=1$ and make 4th ratio,

Then $$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{1\,dx + 1\,dy + 1\,dz}{x(y-z) + y(z-x) + z(x-y)}$$

Now make a group by taking 1st & 4th ratio

$$\Rightarrow \frac{dx}{x(y-z)} = \frac{dx + dy + dz}{x(y-z) + y(z-x) + z(x-y)}$$

$\Rightarrow dx + dy + dz = 0$ $\left(\because x(y-z) + y(z-x) + z(x-y) \right.$
$\Rightarrow xy - xz + yz - yx + zx - yz$
$\left. = 0 \right)$

Now Integrating both sides.

$\Rightarrow \int dx + \int dy + \int dz = \int 0$

$\Rightarrow x + y + z = c_1$ —— (II)

Again choose another set of Multipliers.
i.e. $l' = \frac{1}{x}$, $m' = \frac{1}{y}$, $n' = \frac{1}{z}$ and make
4th ratio from eqⁿ (1)

Then $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)} = \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z) + (z-x) + (x-y)}$

Now make a group by taking 3rd & 4th ratio

$\Rightarrow \dfrac{dz}{z(x-y)} = \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z) + (z-x) + (x-y)}$

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$ $\left(\because y - z + z - x + x - y \right.$
$\left. = 0 \right)$

Now Integrating both sides.

$$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = c_2$$

$$\Rightarrow \log x + \log y + \log z = c_2$$

$$\Rightarrow \log(xyz) = c_2$$

$$\Rightarrow xyz = e^{c_2} = c_3 \quad \text{—— (III)}$$

So the complete sol^n is

$$f(x+y+z,\ xyz) = 0$$

Q:-2 Solve $x(y^2-z^2)p + y(z^2-x^2)q - (x^2-y^2)z = 0$

Given P.D.E is

$$x(y^2-z^2)p + y(z^2-x^2)q = (x^2-y^2)z$$

where $P = x(y^2-z^2)$

$Q = y(z^2-x^2)$

$R = z(x^2-y^2)$

Then A.E is

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} \quad \text{—— (I)}$$

Choose multipliers $l = \frac{1}{x}$, $m = \frac{1}{y}$ & $n = \frac{1}{z}$ and make another ratio from eq^n (I)

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{(y^2-z^2)+(z^2-x^2)+(x^2-y^2)}$$

Choose 1st & 4th ratio, make a group.

$$\Rightarrow \frac{dx}{x(y^2-z^2)} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{(y^2-z^2)+(z^2-x^2)+(x^2-y^2)}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz = 0 \quad \left(\because y^2-z^2+z^2-x^2+x^2-y^2 = 0\right)$$

Integrating both sides,

$$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = c_1$$

$$\Rightarrow \log x + \log y + \log z = c_1$$

$$\Rightarrow \log(xyz) = c_1$$

$$\Rightarrow xyz = c_2 \quad \text{——— (II)}$$

choose another set of multipliers.

let $l' = x$, $m' = y$, $n' = z$ then 4th ratio of A.F. will be.

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} = \frac{x\,dx+y\,dy+z\,dz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)}$$

then taking 3rd & 4th ratio

$$\Rightarrow \frac{dz}{z(x^2-y^2)} = \frac{x\,dx+y\,dy+z\,dz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)}$$

$$\Rightarrow x\,dx+y\,dy+z\,dz=0 \quad (\because x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)=0)$$

Integrating,

$$\Rightarrow \int x\,dx+\int y\,dy+\int z\,dz = C$$

$$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = C$$

$$\Rightarrow x^2+y^2+z^2 = 2C = C_3 \quad \text{—— (III)}$$

So the complete soln is

$$f(xyz,\ x^2+y^2+z^2) = 0$$

LAPLACE TRASFORMS

Let $f(t)$ be a function of 't'.

Then the Laplace Transforms of $f(t)$ is denoted as $L\{f(t)\}$, is defined as.

$$\boxed{L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.}$$

provided integration exists.

* $L\{f(t)\} = F(s)$, s is a parameter may be real or complex.

Laplace Transforms of some elementary functions

(1) $L\{1\} = \frac{1}{s}$

(2) $L\{t^n\} = \frac{n!}{s^{n+1}}$, $n =$ integer.

(3) $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$, n is fraction.

(4) $L\{e^{at}\} = \frac{1}{s-a}$

(5) $L\{\sin at\} = \frac{a}{s^2+a^2}$

(6) $L\{\cos at\} = \frac{s}{s^2+a^2}$

(7) $L\{\sin hat\} = \frac{a}{s^2-a^2}$

(8) $L\{\cos hat\} = \frac{s}{s^2-a^2}$

Proof-(i):-

$\underline{\underline{L.H.S}} \Rightarrow L\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left\{\frac{1}{e^{st}}\right\}_0^\infty$$

$$= -\frac{1}{s}(0-1) \qquad \left[\because \left[\frac{1}{e^{st}}\right]_0^\infty = \frac{1}{e^\infty} - \frac{1}{e^0} = \frac{1}{\infty} - \frac{1}{1} = 0-1 = -1\right]$$

$$= \frac{1}{s} = (R \cdot H \cdot S)$$

Proof(ii) :-

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

L·H·S $\quad L\{t\}$

$$= \int_0^\infty e^{-st} \cdot t \, dt$$

$$= \left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^\infty$$

$$= \left[0 - \frac{1}{s^2}(0-1) \right]$$

$$= \frac{1}{s^2} = R \cdot H \cdot S \text{ (proved)}.$$

Proof (iii) :-

$$L\{e^{at}\} = \frac{1}{s-a}$$

L·H·S $\quad L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} \, dt$

$$= \int_0^\infty e^{-st+at} \, dt = \int_0^\infty e^{-(s-a)t} \, dt.$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{-(s-a)}\left[0-1\right]$$

$$= \frac{1}{s-a} = R \cdot H \cdot S$$

Properties of Laplace Transformation

(1) Property of Linearity :-

$$L\{af(t) + bg(t) - ch(t)\}$$

$$= aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$$

Q:-1 find the Laplace Transforms of.

(i) $e^{2t} + 4t^4 - 3\cos 3t$

(ii) $(\sin t - \cos t)^2$

(iii) $\sin t \cdot \cos 2t$.

(iv) $\cos^2 3t$.

Solution :-

(i) $L\{e^{2t} + 4t^4 - 3\cos 3t\}$

$$= L\{e^{2t}\} + 4L\{t^4\} - 3L\{\cos 3t\}$$

$$= \frac{1}{s-2} + 4\frac{4!}{s^5} - 3\frac{s}{s^2+(3)^2}$$

$$= \frac{1}{s-2} + 4\frac{24}{s^5} - \frac{3s}{s^2+9}$$

(ii) $L\{\sin t - \cos t\}^2$

$$= L\{\sin^2 t + \cos^2 t - 2\sin t \cdot \cos t\}$$

$$= L\{1 - 2\sin t \cdot \cos t\}$$

$$\left(\because 2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)\right)$$

$$= L\{1\} - 2\,L\{\sin t \cdot \cos t\}$$

$$= \frac{1}{s} - 2L\left\{\frac{1}{2}\left(\sin(2t) + \sin(0)\right)\right\}$$

$$= \frac{1}{s} - 2 \times \frac{1}{2}\,L\{\sin 2t\}$$

$$= \frac{1}{s} - \frac{2}{s^2+(2)^2}$$

$$= \frac{1}{s} - \frac{2}{s^2+4} = \frac{s^2+4-2s}{s(s^2+4)}$$

(iii) $L\{\sin t \cdot \cos 2t\}$

$$= L\left\{\frac{1}{2}(2\sin t \cdot \cos 2t)\right\}$$

$$= L\left\{\frac{1}{2}\left(\sin 3t + \sin(-t)\right)\right\}$$

$$= L\left\{\frac{1}{2}(\sin 3t - \sin t)\right\}$$

$$= \frac{1}{2}\left[L\{\sin 3t\} - L\{\sin t\}\right]$$

$$= \frac{1}{2}\left(\frac{3}{s^2+9} - \frac{1}{s^2+1}\right)$$

(iv) $L\{\cos^2 3t\}$.

$$= L\left\{\frac{1}{2}(2\cos^2 3t)\right\} \qquad (\because 2\cos^2\theta = 1+\cos 2\theta)$$

$$= L\left\{\frac{1}{2}\left(1+\cos 6t\right)\right\}$$

$$= \frac{1}{2}\left[L\{1\} + L\{\cos 6t\}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+36}\right)$$

(2) ~~Fist~~ First shifting property :-

$$\boxed{L\{e^{at} f(t)\} = F(s-a)}$$

so by applying this Property , we will get the following formulas.

(1) $L(e^{at}) = \dfrac{1}{s-a}$

(2) $L(e^{at}\, t^{n}) = \dfrac{n!}{(s-a)^{n+1}}$

(3) $L(e^{at} \sin bt) = \dfrac{b}{(s-a)^2 + b^2}$

(4) $L\{e^{at} . \cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}$

(5) $L(e^{at} \sinh bt) = \dfrac{b}{(s-a)^2 - b^2}$

(6) $L\{e^{at} . \cosh bt) = \dfrac{(s-a)}{(s-a)^2 - b^2}$

Q.2 Evaluate the Laplace transforms of

(i) $e^{3t} \cos 4t$

(ii) $e^{2t}\, t^2$

(iii) $e^{-t} \sin^2 t$

(iv) $e^{-2t} \sinh 3t$.

Solutions

(i) $L\{e^{3t} \cos 4t\}$

$=$ $\qquad L\{\cos 4t\} = \dfrac{s}{s^2+(4)^2}$

Then by first shifting property.

$$L\{e^{3t} \cos 4t\} = \frac{(s-3)}{(s-3)^2+(4)^2}$$

$$= \frac{(s-3)}{s^2+9-6s+16}$$

$$= \frac{s-3}{s^2-6s+25}$$

(ii) $L\{e^{2t}\, t^2\}$

$$L\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

Then by first shifting Property,

$$L\{e^{2t}\, t^2\} = \frac{2}{(s-2)^3}$$

(iii) $L\{e^{-t} \sin^2 t\}$

$L\{\sin^2 t\}$

$$= L\left\{\frac{1}{2}(2\sin^2 t)\right\}$$

$$= L\left\{\frac{1}{2}(1-\cos 2t)\right\}$$

$$= L\left\{\frac{1}{2}(1-\cos 2t)\right\}$$

$$= \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Then by first shifting Property.

$L\{e^{-t} \sin^2 t\}$

$$= \frac{1}{2}\left[\frac{1}{s+1} - \frac{(s+1)}{(s+1)^2+4}\right] \text{ (Ans)}.$$

$$= \frac{1}{2}\left[\frac{(s^2+2s+5)-(s+1)}{(s+1)(s^2+2s+5)}\right]$$

(Ans).

* Transform of a function Multiplied by t^n.

i.e. $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

where $L\{f(t)\} = F(s)$

Q:-1 Evaluate L.T. of followings.

(i) $L\{t \cos at\}$

Solution

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{t \cos at\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= (-1)\left[\frac{(s^2+a^2)\frac{d}{ds}s - s\frac{d}{ds}(s^2+a^2)}{(s^2+a^2)^2}\right]$$

$$= (-1)\left[\frac{(s^2+a^2) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= (-1)\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= (-1)\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

(iii) $L\{t^2 e^{-3t}\}$

Solution $L\{e^{-3t}\}$

$$= \frac{1}{s+3}$$

$$L\{t^2 e^{-3t}\}$$

$$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)$$

$$= \frac{d}{ds}\left(\frac{d}{ds}\frac{1}{s+3}\right)$$

$$= \frac{d}{ds}\left(-\frac{1}{(s+3)^2}\right)$$

$$= \frac{-(-2)}{(s+3)^3}$$

$$= \frac{2}{(s+3)^3} \text{ (Ans).}$$

OR $L\{t^2\}$

$$= \frac{2!}{s^{2+1}}$$

$$= \frac{2}{s^3}$$

Then $L\{e^{-3t}\, t^2\}$

$$= \frac{2}{(s+3)^3} \text{ (Ans)}$$

(iii) $L\{t\,e^{-t}\sin 3t\}$

Solution:- $L\{\sin 3t\} = \dfrac{3}{s^2+a^2} = \dfrac{3}{s^2+3^2} = \dfrac{3}{s^2+9}$

$$L\{t\sin 3t\} = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= (-1)\left[\frac{(s^2+9)\left(\frac{d}{ds}3\right) - 3\left(\frac{d}{ds}s^2+9\right)}{(s^2+9)^2}\right]$$

$$= (-1)\left[\frac{(s^2+9)\cdot 0 - 3(2s)}{(s^2+9)^2}\right]$$

$$= (-1)\frac{(-6s)}{(s^2+9)^2}$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L\{e^{-t}\,t\sin 3t\} = \frac{6(s+1)}{[(s+1)^2+9]^2}$$

* Transforms of a function divided by t :

i.e. $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

where $F(s) = L\{f(t)\}$

Q:-1 Evaluate L.T. of the followings.

(i) $L\left\{\frac{1-e^t}{t}\right\}$

solution :- $L\{1-e^t\} = \frac{1}{s} - \frac{1}{s-1}$

$$L\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds.$$

$$= \Big[\log s - \log(s-1)\Big]_s^\infty$$

$$= \left[\log\left(\frac{s}{s-1}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1}{1-1/s}\right)\right]_s^\infty$$

$$= \log 1 - \log\left(\frac{s}{s-1}\right)$$

$$= 0 - \log\left(\frac{s}{s-1}\right) = \log\left(\frac{s-1}{s}\right) \text{ (Ans)}$$

Transforms of derivatives

$$L\{f^{(n)}t\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) \cdots$$

i.e. $L[f'(t)] = s\,L\{f(t)\} - f(0)$

$$L\{f''(t)\} = s^2 L\{f(t)\} - s\,f(0) - f'(0)$$

$$L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - s f'(0) - f''(0)$$

and so on.

Transforms of Integrals

$$\boxed{L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}}$$

where $F(s) = L\{f(t)\}$.

Q:-1 Evaluate L.T. of the followings.

(i) $\int_0^t e^{-3t} \cos t \, dt$:

solution $L\{\cos t\} = \frac{s}{s^2+1}$

$$L\{e^{-3t}\cos t\} = \frac{s+3}{(s+3)^2+1}$$

$$L\left\{\int_0^t e^{-3t}\cos t\, dt\right\} = \frac{s+3}{s\left[(s+3)^2+1\right]}$$

(ii) $\int_0^t \left(\frac{\sin t}{t}\right) dt$.

solution $L\{\sin t\} = \frac{1}{s^2+1}$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\, ds.$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$L\left[\int_0^t \left(\frac{\sin t}{t}\right) dt\right] = \frac{\cot^{-1} s}{s} \quad \text{(Ans)}.$$

Gamma function

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} \, dx .$$

Properties

① $\Gamma(n+1) = n\Gamma(n)$

or $\Gamma(n+1) = n!$ for n is a positive integer

Ex:- $\Gamma(4) = \Gamma(3+1)$

$= 3\,\Gamma(3)$

$= 3\,\Gamma(2+1)$

$= 3\cdot 2\,\Gamma(2)$

$= 3\cdot 2\,\Gamma(1+1)$

$= 3\cdot 2\cdot 1\,\Gamma(1).$

$= 3\cdot 2\cdot 1 = 3!$

Gamma function for n = fractional number.

① $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right)$

$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{1}{2}\cdot\sqrt{\pi}$

NOTE

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

② $\Gamma\left(\frac{9}{2}\right) = \Gamma\left(\frac{7}{2}+1\right)$

$= \frac{7}{2}\Gamma\left(\frac{7}{2}\right)$

$= \frac{7}{2}\cdot\Gamma\left(\frac{5}{2}+1\right)$

$= \frac{7}{2}\cdot\frac{5}{2}\Gamma\left(5/2\right)$

$= \frac{7}{2}\cdot\frac{5}{2}\Gamma\left(\frac{3}{2}+1\right)$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{1}{2}+1\right)$

$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(1/2\right)$

$= \frac{105}{16}\sqrt{\pi}$ (Ans)

Q:- Evaluate the followings.

(i) $L\{t^{3/2}\}$

Solⁿ $L\{t^{3/2}\} = \dfrac{\Gamma(3/2+1)}{s^{3/2+1}} = \dfrac{3/2\,\Gamma(3/2)}{s^{5/2}}$

$$= \frac{\frac{3}{2}\Gamma(1/2+1)}{s^{5/2}} = \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{s^{5/2}}$$

$$= \frac{\frac{3}{4}\sqrt{\pi}}{s^{5/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}} \text{ (Ans)}.$$

(ii) $\cancel{L\left\{\frac{1}{\sqrt{t}}\right\}}$ $L\{\sqrt{t}\}$

Solution:- $L\{t^{1/2}\} = \dfrac{\Gamma(\frac{1}{2}+1)}{s^{1/2+1}} = \dfrac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{1/2+1}}$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}} \text{ (Ans)}.$$

(i) Evaluate: $\Gamma\left(-\frac{1}{2}\right)$

use formula.

Sol^n

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-1/2}$$

$$\Rightarrow \boxed{\Gamma(n) = \frac{\Gamma(n+1)}{n}}$$

$$= -2\,\Gamma\left(\frac{1}{2}\right)$$

$$= -2\sqrt{\pi} \qquad \text{as } \boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

(ii) Evaluate $\Gamma(-7/2)$

Sol^n :- $\Gamma\left(-\frac{7}{2}\right) = \dfrac{\Gamma(-7/2+1)}{-7/2}$

$$= \frac{\Gamma(-5/2)}{-7/2}$$

$$= \cancel{\frac{\Gamma(-5/2+1)}{-7/2 \cdot -5/2}}$$

$$= -\frac{2}{7}\,\frac{\Gamma(-5/2+1)}{-5/2}$$

$$= -2/7 \cdot (-2/5)\;\Gamma(-3/2)$$

$$= \frac{4}{35} \frac{\Gamma(-3/2+1)}{-3/2}$$

$$= \frac{4}{35} \left(-\frac{2}{3}\right) \Gamma(-1/2)$$

$$= \frac{4}{35} \left(-\frac{2}{3}\right) \left(-2\sqrt{\pi}\right)$$

$$= \frac{8 \cdot 2}{105} \sqrt{\pi} = \frac{16}{105} \sqrt{\pi}.$$

(iii) Evaluate. $L\left(\frac{1}{\sqrt{t}}\right)$

Sol^n $L\left(\frac{1}{\sqrt{t}}\right)$

$$= L\left(t^{-1/2}\right)$$

$$= \left\{ \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} \right\}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} \text{ (Ans)}.$$

Inverse Laplace Transformation:-

Inverse Laplace Transformation is the inverse function of laplace Transformation.

i.e. $L\{f(t)\} = F(s)$

Then $\boxed{f(t) = L^{-1} F(s)}$

Then we have the following results from the Laplace Transforms:-

① $L^{-1}\left\{\frac{1}{s}\right\} = 1$

② $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

③ $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

④ $L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

$\Rightarrow L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$ or $\boxed{L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}}$

(5) $L^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$

(6) $L^{-1}\left\{\dfrac{s}{s^2-a^2}\right\} = \cosh at.$

(7) $L^{-1}\left\{\dfrac{a}{s^2-a^2}\right\} = \operatorname{sen} h\, at$

$$\Rightarrow \boxed{L^{-1}\left\{\dfrac{1}{s^2-a^2}\right\} = \dfrac{1}{a}\operatorname{sen} h\, at}$$

(8) $L^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \operatorname{sen} at.$

$$\Rightarrow \boxed{L^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{1}{a}\operatorname{sen} at}$$

(9) $L^{-1}\left\{\dfrac{s+b}{(s+b)^2+a^2}\right\} = e^{-bt}\cos at$

(10) $L^{-1}\left\{\dfrac{1}{(s+b)^2+a^2}\right\} = \dfrac{1}{a}e^{-bt}\operatorname{sen} at.$

(11) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\} = \dfrac{1}{2a}\, t\operatorname{sen} at.$

(12) $L^{-1}\left\{\dfrac{1}{(s^2+a^2)^2}\right\} = \dfrac{1}{2a^3}\left(\operatorname{sen} at - at\cos at\right)$

Q:-1 Evaluate inverse Laplace Transform of the following functions:

(i) $\frac{2s}{s^2+9}$ (ii) $\frac{3s-4}{s^2-25}$ (iii) $\frac{1}{s^3}$

(iv) $\frac{1}{(s+5)^3}$ (v) $\frac{25}{s-6}$

Solution:-

(i) $L^{-1}\left\{\frac{2s}{s^2+9}\right\}$

$= 2\,L^{-1}\left\{\frac{s}{s^2+9}\right\}$

$= 2\cos 3t$

(ii) $L^{-1}\left\{\frac{3s-4}{s^2-25}\right\}$

$= L^{-1}\left\{\frac{3s}{s^2-25}\right\} - 4\,L^{-1}\left\{\frac{1}{s^2-25}\right\}$

$= 3\cosh 5t - 4\cdot\frac{1}{5}\sinh 5t.$

(iii) $L^{-1}\left\{\frac{1}{s^3}\right\}$

$$= \cancel{\frac{(3-1)!}{t^{3-1}}} \qquad \frac{t^{3-1}}{(3-1)!}$$

$$= \cancel{\frac{2!}{t^2}} = \cancel{\frac{2}{t^2}} \qquad = \frac{t^2}{2!}$$

(iv) $L^{-1}\left\{\frac{1}{(s+5)^3}\right\}$

$$= e^{-5t} \cdot \frac{t^2}{2!}$$

(v) $L^{-1}\left\{\frac{25}{s-6}\right\}$

$$= 25\, L^{-1}\left\{\frac{1}{s-6}\right\}$$

$$= 25\, e^{6t}$$

Partial Fraction :-

Case I :- when denominator contains non-repeated linear factors.

$$\frac{P(x)}{(x-a)(x+b)(x-c)} = \frac{A}{x-a} + \frac{B}{x+b} + \frac{C}{x-c}$$

Case II :- when denominator contains repeated linear factors.

$$\frac{P(x)}{(x-a)^2(x+b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x+b}.$$

Case III :- when denominator contains non-repeated quadratic factors which can't be factorised.

$$\frac{P(x)}{(x^2+a)(x^2+c)} = \frac{Ax+B}{x^2+a} + \frac{Cx+D}{x^2+c}.$$

If the quadractic term is factorised then it converted into the linear factors.

In some cases it is not possible to calculate inverse L.T. as it is not in standard form, In that case $f(t)$ can be obtained by resolving $F(s)$ into partial fraction.

Q.1 $L^{-1}\left(\frac{s}{(s-3)(s^2+4)}\right)$

Sol^n consider

$$\Rightarrow \frac{s}{(s-3)(s^2+4)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+4} \quad \text{——} (1)$$

$$= \frac{A(s^2+4)+(Bs+C)(s-3)}{(s-3)(s^2+4)}$$

$$\Rightarrow s = A(s^2+4)+(Bs+C)(s-3)$$

$$\Rightarrow s = As^2+4A+Bs^2-3Bs+Cs-3C$$

$$= (A+B)s^2+(C-3B)s+4A-3C$$

Comparing the coefficients of s^2, s and consts.

$A + B = 0$ — (II)

$C - 3B = 1$ — (III)

$4A - 3C = 0$ — (IV)

we have

$A + B = 0$

$\Rightarrow A = -B$

put in eqn (III)

$C - 3(-A) = 1$

$\Rightarrow C + 3A = 1$

Now $3A + C = 1$

$4A - 3C = 0$ solving these two eqns.

$\therefore A = \frac{3}{13}, \quad C = \frac{4}{13}$

and $B = -A \Rightarrow B = \frac{-3}{13}$

putting the values in eqn (I)

$$\Rightarrow \frac{s}{(s-3)(s^2+4)} = \frac{3/13}{s-3} + \frac{(-3/13)s + 4/13}{s^2+4}$$

Taking inverse L.T. on both sides.

$$\Rightarrow L^{-1}\left\{\frac{s}{(s-3)(s^2+4)}\right\} = L^{-1}\left\{\frac{3/13}{s-3}\right\}$$

$$+ L^{-1}\left\{\frac{-3/13\, s}{s^2+4}\right\} + L^{-1}\left\{\frac{4/13}{s^2+4}\right\}$$

$$= 3/13\, L^{-1}\left\{\frac{1}{s-3}\right\} - 3/13\, L^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$+ 4/13\, L^{-1}\left\{\frac{1}{s^2+4}\right\}.$$

$$= 3/13\, e^{3t} - 3/13 \cos 2t + 4/13 \cdot \frac{1}{2} \sin 2t$$

(Ans).

Q:2 $L^{-1}\left\{\frac{3s}{s^2+2s-8}\right\}$

Sol^n $L^{-1}\left\{\frac{3s}{s^2+4s-2s-8}\right\}$

$$= L^{-1}\left\{\frac{3s}{s(s+4)-2(s+4)}\right\}$$

$$= L^{-1}\left\{\frac{3s}{(s+4)(s-2)}\right\}$$

Now.

$$\left\{\frac{3s}{(s+4)(s-2)}\right\} = \frac{A}{s+4} + \frac{B}{s-2} \quad \text{—— (I)}$$

$$= \frac{A(s-2) + B(s+4)}{(s+4)(s-2)}$$

$$\Rightarrow 3s = A(s-2) + B(s+4)$$

$$= As - 2A + Bs + 4B$$

$$= (A+B)s + 4B - 2A$$

Comparing the coefficients of s and constants.

$$A + B = 3 \quad \text{—— (II)}$$

$$4B - 2A = 0 \quad \text{—— (III)}$$

Solving (II) & (III)

$$2A + 2B = 6$$

$$-2A + 4B = 0$$

$$6B = 6$$

$$\boxed{B = 1}$$

Then $\boxed{A = 2}$

Putting these values in eqn (I)

$$\Rightarrow \frac{3s}{(s+4)(s-2)} = \frac{2}{s+4} + \frac{1}{s-2}$$

Taking I.L.T. on both sides.

$$\Rightarrow L^{-1}\left\{\frac{3s}{s^2+2s-8}\right\} = L^{-1}\left\{\frac{2}{s+4}\right\} + L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 2e^{-4t} + e^{2t} \quad \text{(Ans)}.$$

Q.3 $L^{-1}\left\{\dfrac{s+2}{s^2-4s+13}\right\}$

consider $\dfrac{s+2}{s^2-4s+13}$

[as we can't break the middle term so make perfect square]

$$= \frac{s+2}{s^2-2\cdot s\cdot 2+(2)^2-(2)^2+13}$$

$$= \frac{s+2}{(s-2)^2+9}$$

$$= \frac{s-2+4}{(s-2)^2+9}$$

$$\frac{s+2}{s^2-4s+13} = \frac{s-2}{(s-2)^2+9} + \frac{4}{(s-2)^2+9}$$

Taking I.L.T. on both sides.

$$\Rightarrow L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s-2}{(s-2)^2+9}\right\} + 4\,L^{-1}\left\{\frac{1}{(s-2)^2+9}\right\}$$

$$= e^{2t}\cos 3t + 4\cdot\frac{1}{3}e^{2t}\sin 3t.$$

Solution of linear differential eq^{n}s using L.T.

Q.1 :- Solve $y'' + 5y' + 6y = e^{2x}$, $y(0) = 2$, $y'(0) = -1$

Solution :- Given $y'' + 5y' + 6y = e^{2x}$

Taking L.T. on bothsides.

$$L\{y''(x) + 5y'(x) + 6y(x)\} = L\{e^{2x}\}.$$

$$\Rightarrow L\{y''(x)\} + 5L\{y'(x)\} + 6L\{y(x)\} = \frac{1}{s-2}$$

$$\Rightarrow \left[s^2L\{y(x)\} - sy(0) - y'(0)\right] + 5\left[sL\{y(x)\} - y(0)\right] + 6L\{y(x)\} = \frac{1}{s-2}$$

$$\Rightarrow s^2L\{y(x)\} - 2s - (-1) + 5sL\{y(x) - 5(-1) + 6L\{y(x)\} = \frac{1}{s-2}$$

$$\Rightarrow \{s^2 + 5s + 6\}L\{y(x)\} - 2s + 1 + 5 = \frac{1}{s-2}$$

$$\Rightarrow (s^2 + 5s + 6)L\{y(x)\} = \frac{1}{s-2} + 2s - 6$$

$$\Rightarrow (s^2+5s+6)\, L\{y(x)\} = \frac{1+(2s-6)(s-2)}{s-2}$$

$$= \frac{1+2s^2-4s-6s+12}{s-2}$$

$$\Rightarrow \quad L\{y(x)\} = \frac{2s^2-10s+13}{(s-2)(s^2+5s+6)}$$

$$= \frac{2s^2-10s+13}{(s-2)(s+2)(s+3)}$$

$$\Rightarrow y(x) = L^{-1}\left\{\frac{2s^2-10s+13}{(s-2)(s+2)(s+3)}\right\}$$

Considere $\dfrac{2s^2-10s+13}{(s-2)(s+2)(s+3)} = \dfrac{A}{s-2} + \dfrac{B}{s+2} + \dfrac{C}{s+3}$ —— (I)

$$= \frac{A(s+2)(s+3)+B(s-2)(s+3)+C(s-2)(s+2)}{(s-2)(s+2)(s+3)}$$

$$\Rightarrow 2s^2-10s+13 = A(s+2)(s+3)+B(s-2)(s+3)+C(s-2)(s+2) \quad \text{—— (II)}$$

Expand eqn (II)

$$= A(s^2+3s+2s+6)+B(s^2+3s-2s-6)+C(s^2-4)$$

$$= As^2+5As+6A+Bs^2+Bs-6B+Cs^2-4C$$

$$= (A+B+C)s^2+(5A+B)s+6A-6B-4C$$

Comparing both sides.

$$A+B+C=2 \quad \text{—} \quad (I)$$

$$5A+B=-10 \quad (II) \Rightarrow B=-10-5A$$

$$6A-6B\;4C=13 \quad \text{—} \quad (III)$$

putting the values in eq^n (I) & (III)

$$A-10-5A+C=2 \Rightarrow -4A+C=12 \quad (IV)$$

$$6A-6(-10-5A)-4C=13 \Rightarrow 6A+60+30A-4C=13$$

$$\Rightarrow 36A-4C=-47 \quad (V)$$

solving (IV) & (V)

$$\begin{aligned} -16A+4C &= 48 \\ 36A-4C &= -47 \\ \hline 20A &= 1 \end{aligned}$$

$$\boxed{A=1/20}$$

$$C=\frac{61}{5}$$

$$B=\frac{-41}{4}$$

putting in eq^n (I)

$$\Rightarrow \frac{2s^2-10s+13}{(s-2)(s+2)(s+3)} = \frac{1/20}{s-2} + \frac{(-41/4)}{s+2} + \frac{61/5}{s+3}$$

Taking Inverse L.T. on both sides.

$$\Rightarrow L^{-1}\left\{\frac{2s^2-10s+13}{(s-2)(s+2)(s+3)}\right\} = \frac{1}{20}L^{-1}\left\{\frac{1}{s-2}\right\} - \frac{41}{4}L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{61}{5}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{20}e^{2t} - \frac{41}{4}e^{-2t} + \frac{61}{5}e^{-3t}.$$

(Ans).

Fourier Series :-

Def^n:- Let $f(x)$ be a periodic function with period 2π. Then the trigonometric series.

i.e. $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

is called fourier series where a_0, a_n, b_n are the fourier co-efficient of $f(x)$.

Period function

A function $f : R \longrightarrow R$ is said to be a periodic function of period $\underline{\lambda}$ (a +ve no.) if

$$f(x+\lambda) = f(x)$$

Example $f(x) = \sin x$.

Now $\sin(x+2\pi) = \sin 2\pi$

So $\sin x$ is a periodic function of period 2π.

* Fourier Series is represented in the form of Eulers formulae.

Eulers formulae

The fourier Series for the function $f(x)$ in the interval $(\alpha, \alpha+2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Where

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\,dx.$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\,dx.$$

Now these values a_0, a_n, b_n are known as Euler's formulae.

NOTE-1

(1) If $\alpha = 0$ then ~~interval~~ interval becomes $(0, 2\pi)$

Then

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx.$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx.$$

(2) If $\alpha = -\pi$, then the interval becomes $(-\pi, \pi)$.

Then

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx. \quad \text{or } a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \text{ or } a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \text{ or } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx.$$

NOTE-2 If the given interval is $(-\pi, \pi)$ then check

① If $f(x)$ is an even function, then $b_n = 0$

& $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) dx$

② If $f(x)$ is an odd function, then $a_0 = 0$

& $a_n = 0$ &

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

Q.1 find fourier expansion of $f(x) = x$

$-\pi < x < \pi$

Sol^n Here $f(x) = x$ is an odd function, so

$a_0 = 0$ & $a_n = 0$

$$\& \ b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right\} - \left\{\frac{0}{n} + \frac{\sin 0}{n^2}\right\}\right]$$

$$= \frac{2}{\cancel{\pi}} \left[\left\{ \frac{-\cancel{\pi}(-1)^n}{n} + 0 \right\} - 0 \right]$$

$$\int uv\,dx = u\int v\,dx - \left(\frac{d}{dx}u\right)\iint v\,dx - \left(\frac{d^2}{dx^2}u\right)\iiint v\,dx \ldots$$

$$\frac{2}{n}(-1)^{n+1}$$

$\therefore$ fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx.$$

$$= 2\left[\sin x + \left(\frac{-1}{2}\right)\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

(Ans).